I-YEAR B.E./B.TECH. (COMMON TO ALL BRANCHES)
MATRICES AND CALCULUS (MA3151) QUESTION BANK

UNIT – I MATRICES

PART-A

1	**State Cayley Hamilton theorem. Also write uses of Cayley Hamilton theorem.**
	Statement: Every square matrix satisfies its own characteristic equation. The most important applications of Cayley–Hamilton theorem which are related to the calculation of inverse and powers of square matrices.
2	**If A $= \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$, then find $2A^2 - 12A + 10I$.** **(NOV/ DEC 2019)**
	Solution: $\lvert A - \lambda I \rvert = 0$ $\lambda^2 - 6\lambda + 5 = 0$ $A^2 - 6A + 5I = 0$ (By Cayley Hamilton theorem) Hence, $2(A^2 - 6A + 5I) = 0$ $2A^2 - 12A + 10I = 0$
3	**Show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.**
	Solution: $AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & -\cos\theta\sin\theta+\sin\theta\cos\theta \\ -\sin\theta\cos\theta+\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $A^TA = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta+\sin^2\theta & \cos\theta\sin\theta-\cos\theta\sin\theta \\ \cos\theta\sin\theta-\cos\theta\sin\theta & \sin^2\theta+\cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ $\therefore AA^T = A^TA = I \Rightarrow$ A is orthogonal.
4	**Find the constants 'a' &'b' such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 & –2 as its eigen values. (APR/MAY 2018)**
	Solution: Sum of the eigen values = sum of the main diagonal elements $\Rightarrow a + b = 3 - 2 = 1 - - - - - - (1)$ Product of the eigen values = $\lvert A \rvert \Rightarrow (3)(-2) = ab - 4$ $i.e.\ -6 = ab - 4 \Rightarrow ab = -2$ $\therefore b = -2/a$ $sub\ b\ in\ (1)\ a + b = 1 \Rightarrow a + (-2/a) = 1 \Rightarrow a^2 - 2 = a \quad i.e.\ a^2 - a - 2 = 0$ $solving\ a = -1, 2 \Rightarrow b = 2, -1$
5	**If the product of eigen values of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \\ 1 & k & 2 \end{bmatrix}$ is -8, then find the value of k?**
	Solution: The product of the eigen values = $\lvert A \rvert$ $\lvert A \rvert = 1(10 - 2k) - 2(6 - 2) - 1(3k - 5) = -8$ $\Rightarrow 10 - 2k - 8 - 3k + 5 = -8$ $\Rightarrow 7 - 5k = -8$ Hence, $k = 3$
6	**If $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ is an eigen vector of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, find the corresponding eigen value.**

	Solution: Let X = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be the eigen vector of the matrix corresponding to the eigen value λ. The eigen vectors are obtained from the equation $AX = \lambda X$ $$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$ $$\begin{pmatrix} -2+4+3 \\ 2+2+6 \\ -1-4+0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ -\lambda \end{pmatrix}$$ $$\therefore \lambda = 5.$$
7	**If trace and determinant of 2 × 2 is -2 and -35 respectively, then find the eigen values of matrix?**
	Solution: Let λ_1 and λ_2 are two eigen value of the given matrix A Trace of A= sum of the eigen values $$\Rightarrow \lambda_1 + \lambda_2 = -2 \ldots\ldots\ldots\ldots\ldots.. \quad (1)$$ Determinant of A = Product of the eigen values $$\Rightarrow \lambda_1\lambda_2 = -35 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$ Sub (1) $\Rightarrow \lambda_1 = -2 - \lambda_2$ in equation (2) $$(-2-\lambda_2)\lambda_2 = -35$$ $$(-2-\lambda_2)\lambda_2 + 35 = 0$$ $$\lambda_2^2 + 2\lambda_2^2 - 35 = 0$$ $$(\lambda_2 - 5)(\lambda_2 + 7) = 0$$ Therefore, $\lambda_2 = 5$ and -7 If the $\lambda_2 = 5$ then $\lambda_1 = -2 - 5 = -7$ If the $\lambda_2 = -7$ then $\lambda_1 = -2 + 7 = 5$ Hence the eigen values are 5 and -7.
8	**If A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}$ then find the eigen values of A^{-1} and A^2-2I.** **(APR/MAY 2018)**
	Solution: In a triangular matrix, the main diagonal values are the eigen values of the matrix. ∴ 2, 3, 4 are the eigen values of A. Hence the eigen values of A^{-1} = $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. The eigen values of A^2 are $(2)^2, (3)^2, (4)^2$ = 4, 9, 16. The eigen values of A^2-2I are $4-2, 9-2, 16-2 = 2, 7, 14$.
9	**Determine whether the given matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is Diagonalizable?**
	Solution: The given matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ is triangular matrix ∴ the eigenvalues are $\lambda_1 = 2, \lambda_2 = 6$ and $\lambda_3 = 1$. since eigenvectors corresponding to distinct eigenvalues are linearly independent, A has three linearly independent eigenvectors and it is therefore diagonalizable.

10	**If** $\mathbf{A}=\begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$**, then find the eigen values of adjoint of A.** **(APR/MAY 2019)**
	Solution: We know that, adjoint of A = $A^{-1}\lvert A\rvert$. $\lvert A\rvert$ = product of the eigen values = (2)(3)(1) = 6. Eigen values of A^{-1} are $\frac{1}{2},\frac{1}{3},1$. $\therefore$ Eigen values of adj A are $\frac{1}{2}(6),\ \frac{1}{3}(6),\ (1)(6) = 3,\ 2,\ 6$.
11	**Write down the quadratic form corresponding to the matrix** $\mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{5} & \mathbf{-1} \\ \mathbf{5} & \mathbf{1} & \mathbf{6} \\ \mathbf{-1} & \mathbf{6} & \mathbf{2} \end{bmatrix}$**.**
	Solution: The matrix of the quadratic form of A is given by $X^TAX = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0x_1^2 + x_2^2 + 2x_3^2 + 10x_1x_2 + 12x_2x_3 - 2x_1x_3$.
12	**Determine the nature of the following quadratic form** $\boldsymbol{f(x_1,x_2,x_3) = x_1^2 + 2x_2^2 - 9x_3^2}$**.**
	Solution: The matrix of the quadratic form is $Q=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -9 \end{bmatrix}$ The eigen values of the matrix are 1, 2, -9 Therefore, the quadratic form is indefinite.
13	**Discuss the nature of the quadratic form** $\boldsymbol{2x^2 + 3y^2 + 2z^2 + 2xy}$ **without reducing it to canonical form.**
14	**Solution:** The matrix of the quadratic form is $Q=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ The principal sub determinants are $D_1 = 2\ (+ve)$ $D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5(+ve)$ $D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2(6-0) - 1(2-0) + 0 = 10$ Therefore, the quadratic form is positive definite.
14	**Find the values of** $\boldsymbol{\lambda}$ **so that the QF** $\boldsymbol{\lambda(x_1^2 + x_2^2 + x_3^2) + 2x_1x_2 - 2x_2x_3 + 2x_3x_1}$ **may be positive definite.**

Solution: The matrix of the quadratic form is $A = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{vmatrix}$

The principal sub determinants are

$$D_1 = \lambda$$

$$D_2 = \begin{vmatrix} \lambda & 1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$$D_3 = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & -1 \\ 1 & -1 & \lambda \end{vmatrix} = \lambda(\lambda^2-1) - 1(\lambda+1) + 1(-1-\lambda)$$

$$= \lambda^3 - 3\lambda - 2 = (\lambda+1)^2(\lambda-2)$$

For positive definite

$D_1 > 0, D_2 > 0, D_3 > 0$

$\Rightarrow \lambda > 0, \lambda^2 - 1 > 0, (\lambda+1)^2(\lambda-2) > 0$

$\Rightarrow \lambda > 0, (\lambda+1)(\lambda-1) > 0, (\lambda+1)^2(\lambda-2) > 0$

$\Rightarrow \lambda - 2 > 0$

$\Rightarrow \lambda > 2$

Therefore, the quadratic form is positive definite.

15 **Find the matrix whose eigen values are 1,3 and eigen vectors are $(1,-1)^T$, $(1,1)^T$.**

Solution: Since the given eigen vectors $\begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are orthogonal

By orthogonal reduction

$$N^T A N = D$$

$$A = N D N^T$$

The normalized model matrix

$$N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \text{A} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

16 **Find the nature of the conic $8x^2 - 4xy + 5y^2 = 36$ by reducing the quadratic form $8x^2 - 4xy + 5y^2$ to the form $AX^2 + BY^2$.**

Solution: The matrix of the quadratic form is $A = \begin{pmatrix} 8 & -2 \\ -2 & 5 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda^3 - 13\lambda^2 + 36 = 0$$

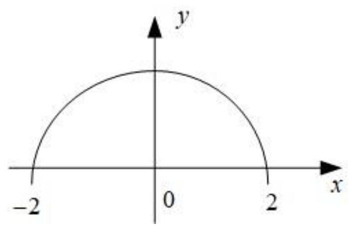
$$\lambda^2 - 13\lambda + 36 = 0$$

$$(\lambda-4)(\lambda-9) = 0$$

$\therefore$ The eigen values of A are $\lambda = 4, 9$

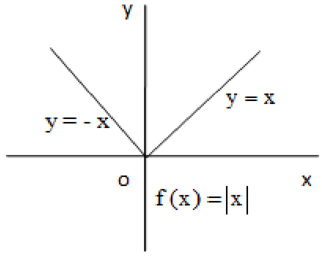
	$\therefore$ the canonical form is of the form $4X^2+9Y^2$ $\therefore$ given conic becomes $4X^2+9Y^2=36$ i.e. $\frac{X^2}{9}+\frac{Y^2}{4}=1$ which is an ellipse.
17	**Write down the matrix equation of an ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is in standard position to the coordinate axis .**
	Solution: The equation of an ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is in quadratic form. Hence $\frac{x^2}{16}+\frac{y^2}{9}=1$ $\Rightarrow [x\ \ y]\begin{bmatrix}\frac{1}{16} & 0\\ 0 & \frac{1}{9}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=1.$
18	**If λ is the eigen value of the matrix A, then prove that λ^2 is the eigen value of A^2. (APR/MAY2019)**
	Solution: If X is the eigen vector of the matrix A corresponding to the eigen value λ, then AX = λX. Pre multiply by A $\Rightarrow A^2 X = A(\lambda X)$ $= \lambda(AX)$ $= \lambda(\lambda X)$ $= \lambda^2 X$ Hence, λ^2 is the eigen value of A^2.
19	**Prove that the matrices A and A^T have the same Eigen values. (NOV/DEC2019)**
	Solution: We know that, $\lvert A\rvert = \lvert A^T\rvert$ **.** $\therefore \lvert A-\lambda I\rvert = \lvert (A-\lambda I)^T\rvert$ $= \lvert (A^T-\lambda I^T)\rvert$ $= \lvert A^T-\lambda I)\rvert \quad (\because I = I^T)$ $\therefore$ A and A^T have same characteristic polynomial $\Rightarrow$ A and A^T have same characteristic equation. Hence, A and A^T have same eigen values.
20	**Show that the eigen values of $-3A^{-1}$ are the same as those of $A=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$**
	Solution: The characteristic equation of A is $\begin{vmatrix}1-\lambda & 2\\ 2 & 1-\lambda\end{vmatrix}=0$ $\lambda^2-2\lambda-3=0$ $(\lambda+1)(\lambda-3)=0$ $\therefore\ \lambda=-1,3$ are the eigen values of A. Now the eigen values of A^{-1} are $\frac{1}{-1},\frac{1}{3}$ and hence the eigen values of $-3A^{-1}$ are 3, –1.
	PART – B
1	**(i) Verify Caley Hamilton theorem for $A=\begin{pmatrix}1 & 1 & 3\\ 1 & 3 & -3\\ -2 & -4 & -4\end{pmatrix}$. Also find adj(A) and A^{-1}.** **(APR / MAY 18)**

	(ii) Verify Cayley-Hamilton theorem for the matrix A= $\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$**, hence find A^4 and A^{-1}.**
2	**(i) Use Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ to express as a linear polynomial in $A^6 - 5A^5 + 8A^4 - 2A^3 - 9A^2 - 31A - 36I$.**
	(ii) Using Cayley-Hamilton theorem, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 - 8A^2 + 2A - I$ **when** $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.
3	**(i) Find the eigen values and eigen vectors of the matrix** $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$
	(ii) Find the eigen values and eigen vectors of the matrix $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$**. Determine the algebraic and geometric multiplicity.**
4	**(i) The Eigen vectors of a 3×3 real symmetric matrix A corresponding to the eigen values 2,3,6 are $(1,0,-1)^T$, $(1,1,1)^T$ and $(1,-2,1)^T$ respectively. Find the matrix A.**
	(ii) Prove that the eigen vectors of the real symmetric matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ **are orthogonal in pairs**
5	**(i) Diagonalize the matrix** $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **by means of orthogonal transformation.**
	(ii) Show that the matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ **is diagonalizable, hence find M such that $M^{-1}AM$ is a diagonal matrix. Then obtain the matrix $B = A^2 + 5A + 3I$**
6	**(i) Compute the eigen values and eigen vectors of the following system:** $10x_1 + 2x_2 + x_3 = \lambda x_1$, $2x_1 + 10x_2 + x_3 = \lambda x_2$, $2x_1 + x_2 + 10x_3 = \lambda x_3$
	(ii) Verify that the eigen vector of the real symmetric matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ **are orthogonal to each other**
7	**Reduce the quadratic form** $10x^2 + 2y^2 + 5z^2 + 6yz - 10xz - 4xy$ **to canonical form by orthogonal reduction. Also find its nature.**

8	**Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to canonical form through orthogonal transformation. Find also its nature. (APR / MAY 18)**
9	**Reduce the quadratic form $2xy+2yz+2xz$ to canonical form by orthogonal reduction. Also find its nature.**
10	**(i) An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2+x_2^2=1$ is stretched so that a point P: (x_1,x_2) goes over into the point Q: (y_1,y_2) given by $\mathbf{y}=\begin{bmatrix}y_1\\y_2\end{bmatrix}=A\mathbf{x}=\begin{bmatrix}5&3\\3&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ in components, $\begin{matrix}y_1=5x_1+3x_2\\y_2=3x_1+5x_2\end{matrix}$ Find the principal directions, that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?**
	(ii) Find the characteristic equation of the matrix $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$ and hence determine its inverse.
11	**An elastic membrane in the x_1x_2-plane with boundary $x_1^2+x_2^2=1$ is stretched so that point $P(x_1,x_2)$ goes over into point $Q(y_1,y_2)$ such that $\mathbf{y}=A\mathbf{x}$ with $A=\frac{3}{2}\begin{bmatrix}2&1\\1&2\end{bmatrix}$. Find the principal directions (Eigen vector) and corresponding factors of extension or contraction of the elastic deformation (eigenvalues). Sketch the shape of the deformed membrane.**
	UNIT II – DIFFERENTIAL CALCULUS
	PART – A
1	**Find the domain of the function $f(x)=\sqrt{3-x}-\sqrt{2+x}$. (NOV/DEC 2018)**
	Solution: The given function is $f(x)=\sqrt{3-x}-\sqrt{2+x}$. Since the square root of a negative number is not defined (as a real number), the domain of $f(x)$ consist of all the values of x such that $3-x\ge 0$ and $x+2\ge 0$. This is equivalent to $x\le 3$ and $x\ge -2$, so the domain of the given function is $[-2,3]$.
2	**Find the domain and range and sketch the graph of the function $f(x)=\sqrt{4-x^2}$**
	Solution: Given $y=\sqrt{4-x^2}$ $\therefore y^2=4-x^2$ $\Rightarrow x^2+y^2=4$ which represents a circle with centre at origin $(0,0)$ and radius 2.  Since $y=\sqrt{4-x^2}\ge 0$, the desired graph is the upper half of the circle. The domain is the interval $-2\le x\le 2$ and the range is the interval $0\le y\le 2$.
3	**Sketch the graph of the absolute value function $f(x)=\begin{cases}x, & x\ge 0\\-x, & x<0\end{cases}$.**
	Solution:

The given function is $f(x)=|x|=\begin{cases} x, & x \ge 0 \\ -x, & x<0 \end{cases}$.

The graph of $f(x)$ coincides with the line $y=x$ to the right of the $y-$axis and coincides with the line $y=-x$ to the left of the $y-$axis.



4 **If $\lim\limits_{x\to1}\dfrac{f(x)-8}{x-1}=10$, then find $\lim\limits_{x\to1} f(x)$** **(NOV/ DEC 2020)**

Solution:

$$\text{Let } \lim_{x\to1} f(x)=\lim_{x\to1}[f(x)+8-8]=\lim_{x\to1}\left[\frac{f(x)-8}{x-1}.(x-1)\right]+8$$

$$=\left[\left[\lim_{x\to1}\frac{f(x)-8}{x-1}\right].\left[\lim_{x\to1}(x-1)\right]\right]+8$$

$$= 0+8=8.$$

5 **Evaluate $\lim\limits_{t\to1}\dfrac{t^4-1}{t^3-1}$.** **(NOV/ DEC 2018)**

Solution:

$$\lim_{t\to1}\frac{t^4-1}{t^3-1}=\lim_{t\to1}\frac{(t^2)^2-1^2}{t^3-1^3}=\lim_{t\to1}\frac{(t^2-1)(t^2+1)}{(t-1)(t^2+t+1)}$$

$$\left[\begin{array}{l} \because a^2-b^2=(a-b)(a+b) \\ a^3-b^3=(a-b)\left(a^2+ab+b^2\right)\end{array}\right]$$

$$=\lim_{t\to1}\frac{\cancel{(t-1)}(t+1)(t^2+1)}{\cancel{(t-1)}(t^2+t+1)}$$

$$=\lim_{t\to1}\frac{(t+1)(t^2+1)}{(t^2+t+1)}=\frac{(1+1)(1^2+1)}{1^2+1+1}=\frac{4}{3}.$$

6 **Check whether $\lim\limits_{x\to3}\dfrac{3x+9}{|x+3|}$ exist.** **(APR/ MAY 2019)**

Solution: $\lim\limits_{x\to3^-}\dfrac{3x+9}{|x+3|}=\lim\limits_{x\to3^-}\dfrac{3x+9}{-(x+3)}=-3$

$$\lim_{x\to3^+}\frac{3x+9}{|x+3|}=\lim_{x\to3^+}\frac{3x+9}{x+3}=3$$

Here $\lim\limits_{x\to3^+}\dfrac{3x+9}{x+3}\neq\lim\limits_{x\to3^-}\dfrac{3x+9}{x+3}$ $\quad\therefore\ \lim\limits_{x\to3}\dfrac{3x+9}{|x+3|}$ does not exist.

7 **If $f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4 \\ 8-2x, & \text{if } x<4\end{cases}$, determine whether $\lim\limits_{x\to4} f(x)$ exists**

Solution:

Given $f(x)=\begin{cases}\sqrt{x-4}, & \text{if } x>4 \\ 8-2x, & \text{if } x<4\end{cases}$

$$\therefore \lim_{x\to4^+} f(x)=\lim_{x\to4^+}\sqrt{x-4}=\sqrt{4-4}=0$$

	$\therefore \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(8-2x) = 8-2(4) = 0$ $\therefore \lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x) = 0$ $\therefore$ the limit exists, and $\lim_{x\to 4} f(x) = 0$.
8	**Show that the function $f(x) = 4-\sqrt{4-x^2}$ is continuous in the interval $[-2,2]$.**
	Solution: If $-2<a<2$, then $\lim_{x\to a} f(x) = \lim_{x\to a}\left(4-\sqrt{4-x^2}\right) = 4-\sqrt{\lim_{x\to a}\left(4-x^2\right)} = 4-\sqrt{4-a^2} = f(a)$ $\therefore f(x)$ is continuous at a if $-2<a<2$. Similarly $\lim_{x\to(-2)^+} f(x) = 4 = f(-2)$ and $\lim_{x\to 2^-} f(x) = 4 = f(2)$ $\therefore f(x)$ is continuous on $[-2,2]$.
9	**Suppose $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) = f(1)$. What are possible values of *a* and *b*.**
	Solution: The given function is $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax, & x>1 \end{cases}$ Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(a+bx) = a+b$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(b-ax) = b-a$ also $f(1) = 4$ It is given that $\lim_{x\to 1} f(x) = f(1)$ $\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$ $\Rightarrow a+b=4$ and $b-a=4$ On solving these two equations, we obtain *a*=0, *b*=4. Thus, the respective possible values of *a* and *b* are 0 and 4.
10	**If $f(x) = xe^x$ then find the expression for $f''(x)$.** **(NOV/DEC 2019)**
	Solution: Given $f(x) = xe^x$ $f'(x) = x(e^x) + e^x(1)$ $\quad [\because (uv)' = uv' + vu']$ $f''(x) = \left[x(e^x) + e^x(1)\right] + e^x$ $f''(x) = x(e^x) + 2e^x = e^x(x+2)$.
11	**If $xe^y = x-y$, then find $\dfrac{dy}{dx}$ by implicit differentiation** **(NOV/DEC 2020)**
	Solution: Given $xe^y = x-y$ Let $f(x,y) = xe^y - x + y$ By formula $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

	$f_x = e^y - 1$ & $f_y = xe^y + 1$ $\therefore \frac{dy}{dx} = -\frac{(e^y - 1)}{xe^y + 1}$ $= \frac{1 - e^y}{(1 + xe^y)}.$
12	**If $f(x) = \sqrt{x}$, then find the derivative of $f(x)$. Also state the domain of $f'(x)$.**
	Solution: $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ $= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$ $\left(\because f(x) = \sqrt{x} \Rightarrow f(x+h) = \sqrt{x+h}\right)$ $= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} . \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$ $= \lim_{h\to 0} \left(\frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \right)$ $= \lim_{h\to 0} \left(\frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} \right) = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$ We see that $f'(x)$ exists only if $x > 0$, so the domain of $f'(x)$ is $(0, \infty)$. This is slightly smaller than the domain of $f(x)$, which is $[0, \infty)$.
13	**Find $\frac{dy}{dx}$ if $\sin(x + y) = y^2 \cos x$**
	Solution: Given $\sin(x + y) = y^2 \cos x$ Diff. w.r.to x, we get, $\cos(x+y)\left(1 + y'\right) = y^2(-\sin x) + (\cos x)\left(2yy'\right)$ $\cos(x+y) + \cos(x+y)y' = -y^2 \sin x + 2yy' \cos x$ $\cos(x+y) + y^2 \sin x = \left(2y\cos x - \cos(x+y)\right)y'$ $\therefore y' = \frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)}.$
14	**Find $\frac{dy}{dx}$ if $x = at^2$, $y = 2at$**
	Solution: Since $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ $x = at^2,\ y = 2at$ $\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a$

	$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}$ $=\frac{2a}{2at}=\frac{1}{t}$
15	**Find the critical points of $y=5x^3-6x$.** **(APR/MAY 2019)**
	Solution: A critical point of a function $y=f(x)$ is a point c in the domain of $f(x)$ such that either $y'(c)=0$ or $y'(c)$ does not exist. $y'=15x^2-6=0.$ $15x^2=6$ $x^2=\frac{6}{15}=\frac{2}{5} \Rightarrow x=\pm\sqrt{\frac{2}{5}}.$
16	**Find an equation of the tangent line to the curve $y=\frac{e^x}{1+x^2}$ at the point $\left(1,\frac{e}{2}\right)$**
	Solution: Given $y=\frac{e^x}{1+x^2}$ $\therefore \frac{dy}{dx}=\frac{(1+x^2)\frac{d}{dx}(e^x)-e^x\frac{d}{dx}(1+x^2)}{(1+x^2)^2}=\frac{(1+x^2)e^x-e^x(2x)}{(1+x^2)^2}=\frac{e^x(1-x)^2}{(1+x^2)^2}$ The slope of the tangent line at $\left(1,\frac{e}{2}\right)$ is $\left.\frac{dy}{dx}\right\|_{x=1}=0$ This means that the tangent line at $\left(1,\frac{e}{2}\right)$ is horizontal and its equation is given by $y-y_1=m(x-x_1)$ $y-\frac{e}{2}=0(x-1)$ $y-\frac{e}{2}=0$ $y=\frac{e}{2}.$
17	**Find the points on the curve $y=x^4-6x^2+4$ where the tangent line is horizontal.**
	Solution: Horizontal tangents occur where the first derivative of the function *y* is zero. Given $y=x^4-6x^2+4$ $\therefore \frac{dy}{dx}=4x^3-12x=0$ $4x(x^2-3)=0$ $x=0,\ x^2-3=0 \Rightarrow x=\pm\sqrt{3}$ Therefore the given curve has horizontal tangents when $x=0,\ x=\sqrt{3},\ x=-\sqrt{3}$. The corresponding points are $(0,4),(\sqrt{3},-5),(-\sqrt{3},-5)$.
18	**Find the equation of the tangent line to the curve $y=x^4+2x^2-x$ at the point (1,2)** **(NOV/DEC 2020)**

	Solution: The equation of tangent line at (x_1, y_1) is given by $(y-y_1)=m(x-x_1)$ Given $y = x^4+2x^2-x$ $\therefore \quad \frac{dy}{dx}=4x^3+4x-1$ at $(1,2)$, $\frac{dy}{dx}=4(1)^3+4(1)-1=7$ $\therefore$ Equation of tangent line is $y-2=7(x-1)$ $\Rightarrow y=7x-5.$
19	**State the extreme value theorem.**
	Solution: If $f(x)$ is continuous on a closed interval $[a,b]$, then $f(x)$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some points c and d in $[a,b]$.
20	**State Fermat's theorem.**
	Solution: If $f(x)$ has a local maximum or minimum at *c*, and if $f'(c)$ exists, then $f'(c)=0$.
	PART – B
1	**(i) A function $f(x)$ is defined by $f(x)=\begin{cases}1-x, & \text{if } x\le -1\\ x^2, & \text{if } x>-1\end{cases}$. Evaluate $f(-2)$, $f(-1)$ and $f(0)$ and sketch the graph.**
	(ii) If $f(x)=\begin{cases}x^2+1, & \text{if } x<1\\ (x-2)^2, & \text{if } x\ge 1\end{cases}$, (i) Find $\lim_{x\to1^-} f(x)$ and $\lim_{x\to1^+} f(x)$, (ii) Does $\lim_{x\to1} f(x)$ exists.
	(iii) Show that $\lim_{x\to0} x^3\sin\left(\frac{1}{x}\right)=0$.
2	**(i) Find the values of *a* and *b* that make $f(x)$ continuous on $(-\infty,\infty)$** $f(x)=\begin{cases}\frac{x^3-8}{x-2}, & \text{if } x<2\\ ax^2-bx+3, & \text{if } 2\le x<3\\ 2x-a+b, & \text{if } x\ge3\end{cases}$. **(NOV/DEC 2018)**
	(ii) For what value of the constant b, is the function $f(x)$ continuous on $(-\infty,\infty)$ if $f(x)=\begin{cases}bx^2+2x, & x<2\\ x^3-bx, & x\ge2\end{cases}$. **(APR/MAY 2019)**
3	**(i) The equation of motion of a particle is $s=t^3-3t$, where *s* is in meters and *t* is in seconds. Find (*a*) the velocity and acceleration as functions of *t*, (*b*) the acceleration after 2*s* and (*c*) the acceleration when the velocity is zero.**
	(ii) Show that the function $f(x)=\lvert x-6\rvert$ is not differentiable at 6. Find a formula for first derivative of *f*(*x*) and sketch its graph. (NOV/DEC 2020)

4	**(i) If $f(x)=\dfrac{1-x}{2+x}$ then, find the equation for $f'(x)$ using the concept of derivatives.** **(NOV/DEC 2019)**
	(ii) Find $\dfrac{dy}{dx}$ if $y=x^2e^{2x}\left(x^2+1\right)^4$. **(APR/MAY 2019)**
5	**(i) Find the derivative of $f(x)=\cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$.** **(NOV/DEC 2018)**
	(ii) Find y' if $\tan(x-y)=\dfrac{y}{1+x^3}$. **(NOV/DEC 2018)**
6	**(i) Find the derivative of $f(x)=\tanh^{-1}\left[\tan\dfrac{x}{2}\right]$.** **(NOV/DEC 2019)**
	(ii) If $f(x)=\dfrac{1-\sec x}{\tan x}$, then find the first derivative of $f(x)$.
	(iii) Differentiate $f(x)=\log_{10}(2+\sin x)$
	(iv) Find $\dfrac{dy}{dx}$ if $y=\sqrt{x}\,\log x$
7	**(i) Use the intermediate value theorem to show that there is a root of the equation $\sqrt[3]{x}=1-x$ in the interval (0,1).** **(NOV/DEC 2020)**
	(ii) Find the absolute maximum and absolute minimum values of $f(x)$ on the given interval $(i)\ f(x)=3x^4-4x^3-12x^2+1,\quad [-2,3]$, $(ii)\ f(x)=2\cos x-\sin 2x,\quad \left[0,\dfrac{\pi}{2}\right]$
8	**For the function $f(x)=2x^3+3x^2-36x$** **(APR/MAY 2019)& (NOV/DEC 2019)** **(i) Find the intervals on which it is increasing and decreasing.** **(ii) Find the local maxima and minima values of $f(x)$.** **(iii) Find the intervals of concavity and the inflection points.**
9	**For the function $f(x)=2+2x^2-x^4$, find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and the inflection points.**
10	**Find the local maximum value and local minimum value, the interval of concavity and the inflection points of a function $f(x)=x^3-3x^2-12x$, Also sketch the graph of that satisfies all the above conditions.** **(NOV/DEC2020)**
	UNIT III – FUNCTIONS OF SEVERAL VARIA BLES
	PART – A
1	**If $u=xy-2yz+z^2$, then find the value of $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}$.**
	Solution: $u=xy-2yz+z^2$ $\dfrac{\partial u}{\partial x}=y;\ \dfrac{\partial u}{\partial y}=x-2z;\ \dfrac{\partial u}{\partial z}=-2y+2z;$ $\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}=y+x-2z-2y+2z=x-y$

2	**If $z = x^2 - 3xy^2$, find the value of $3x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$**
	Solution: Given $z = x^2 - 3xy^2$ $\frac{\partial z}{\partial x} = 2x - 3y^2$; $\frac{\partial z}{\partial y} = -6xy$ $\frac{\partial^2 z}{\partial x^2} = 2;$ $\frac{\partial^2 z}{\partial y^2} = -6x$ $3x\frac{\partial^2 z}{\partial x^2} = 6x$ $3x\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 6x - 6x = 0$
3	**Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u(x,y) = x^y + y^x$** **(NOV/DEC 2020)**
	Solution: Given $u(x,y) = x^y + y^x$ $\frac{\partial u}{\partial x} = yx^{y-1} + y^x \log(y)$ $\left(\because \frac{d}{dx}(a^x) = a^x \log a, \text{ but } \frac{d}{dx}(x^a) = ax^{a-1}\right)$ $\frac{\partial u}{\partial y} = x^y \log(x) + xy^{x-1}$
4	**Verify the Euler's theorem for the function $u = x^2 + y^2 + 2xy$.** **(NOV/DEC 2019)**
	Solution: **Euler's theorem:** If $u(x,y)$ is homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ $u(x,y) = x^2 + y^2 + 2xy$ $u(x,y) = x^2 + y^2 + 2xy \Rightarrow u(tx,ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)$ $u(tx,ty) = t^2x^2 + t^2y^2 + 2t^2xy = t^2\left(x^2 + y^2 + 2xy\right)$ *i.e* t^2u $\therefore$ The degree of $u(x,y) = x^2 + y^2 + 2xy$ is 2. $\therefore$ Degree = n = 2. $\because$ u is homogeneous function of 2nd degree in x and y, by Euler's theorem we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ $(i.e)\, x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(x^2 + y^2 + 2xy\right)$ Now, $u = x^2 + y^2 + 2xy$ $x\frac{\partial u}{\partial x} = x(2x + 2y) = 2x^2 + 2xy$, $y\frac{\partial u}{\partial y} = y(2y + 2x) = 2y^2 + 2xy$ $L.H.S = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2x^2 + 2xy + 2y^2 + 2xy = 2x^2 + 2y^2 + 4xy = 2(x^2 + y^2 + 2xy) = R.H.S$
5	**If $z = xf\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ using Euler's theorem** **(NOV / DEC 2020)**
	Solution: Given $z = xf\left(\frac{y}{x}\right)$.

	i.e., $z(x,y)=xf\left(\frac{y}{x}\right)$ $z(tx,ty)=txf\left(\frac{ty}{tx}\right)=txf\left(\frac{y}{x}\right)=t.z(x,y)$ $z(tx,ty)=t^1 z(x,y)$ $\therefore z(x,y)$ is a homogeneous function of degree 1 in x and y. $\therefore$ Degree = n = 1. **Euler's theorem:** If $u(x,y)$ is homogeneous function of degree n in x and y then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$. $\because$ z is a homogeneous function of 1st degree in x and y, by Euler's theorem , we have $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1(z)=z$
6	**If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$.**
	Solution: $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\Rightarrow \cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}=f\ (x,y)$ $f(x,y)=\frac{x+y}{\sqrt{x}+\sqrt{y}}$ $f(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=\frac{t(x+y)}{\sqrt{t}\left(\sqrt{x}+\sqrt{y}\right)}=\frac{\sqrt{t}(x+y)}{\left(\sqrt{x}+\sqrt{y}\right)}=t^{1/2}f(x,y)$ $f=\cos u$ is a homogeneous function of degree $n=\frac{1}{2}$ in x and y. Therefore, by Euler's theorem we get, $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$ $x\frac{\partial(\cos u)}{\partial x}+y\frac{\partial(\cos u)}{\partial y}=\frac{1}{2}\cos u$ $x(-\sin u)\frac{\partial u}{\partial x}+y(-\sin u)\frac{\partial u}{\partial y}=\frac{1}{2}\cos u$ $-\sin u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=\frac{1}{2}\cos u$ $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\frac{\cos u}{\sin u}=-\frac{1}{2}\cot u$
7	**If $u=x^3+y^3$ and $x=at^2,\ y=2at$, then find $\frac{du}{dt}$.** **(APR/MAY 2019)**
	Solution: Given $u=x^3+y^3$ and $x=at^2,\ y=2at$ $\frac{du}{dt}=\frac{\partial u}{\partial x}.\frac{dx}{dt}+\frac{\partial u}{\partial y}.\frac{dy}{dt}$

$u=x^3+y^3$	$x=at^2$	$y=2at$

$\frac{\partial u}{\partial x}=3x^2$	$\frac{\partial u}{\partial y}=3y^2$	$\frac{dx}{dt}=2at$	$\frac{dy}{dt}=2a$

$$\frac{du}{dt}=(3x^2)(2at)+(3y^2)(2a)$$

$$=(3(at^2)^2)(2at)+(3)(2at)^2(2a) \qquad (\because x=at^2,\ y=2at)$$

$$=6a^3t^5+24a^3t^2=6a^3t^2(t^3+4)$$

8 **Find $\frac{dy}{dx}$, if $x^3+y^3=6xy$.**

Solution:

$$x^3+y^3-6xy=0$$

$$Let\ \ f(x,y)=x^3+y^3-6xy$$

$$f_x=\frac{\partial f}{\partial x}=3x^2-6y;\quad f_y=\frac{\partial f}{\partial y}=3y^2-6x$$

$$\frac{dy}{dx}=\frac{-f_x}{f_y}=\frac{-(3x^2-6y)}{3y^2-6x}=\frac{6y-3x^2}{3y^2-6x}=\frac{2y-x^2}{y^2-2x}$$

9 **What is the derivative of u with respect to x, for $u=x^2y^3$, where $2\sin x-3y=0$?**

Solution:

$$2\sin x-3y=0$$

$$2\cos x-3\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{2\cos x}{3}$$

$$\frac{du}{dx}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$$=2xy^3+3x^2y^2.\frac{2\cos x}{3}$$

$$=2xy^3+2x^2y^2\cos x$$

$$=2xy^2(y+x\cos x)$$

10 **State the properties of Jacobian.** **(NOV/DEC 2018)**

Solution:

(i) If *u* and *v* are functions of *r* and *s*, *r* and *s* are functions of *x* and *y* then, $\frac{\partial(u,v)}{\partial(r,s)}\frac{\partial(r,s)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(x,y)}$

(ii) If *u* and *v* are functions of *x* and *y* then, $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)}=1$ (i.e) $JJ'=1$

(iii) If u, v, w are functionally dependent functions of three independent variable x, y, z then $\frac{\partial(u,v,w)}{\partial(x,y,z)}=0$

11 **If $x=u^2-v^2$ and $y=2uv$, find the Jacobian of x and y with respect to u and v.** **(APR/MAY 2019)**

Solution: $x=u^2-v^2\Rightarrow \frac{\partial x}{\partial u}=2u;\ \frac{\partial x}{\partial v}=-2v$ and $y=2uv\Rightarrow \frac{\partial y}{\partial u}=2v;\ \frac{\partial y}{\partial v}=2u$

	$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$
12	**If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.** **(JAN 2018)**
	Solution: $x = uv \Rightarrow \frac{\partial x}{\partial u} = v;\ \frac{\partial x}{\partial v} = u$ and $y = \frac{u}{v} \Rightarrow \frac{\partial y}{\partial u} = \frac{1}{v};\ \frac{\partial y}{\partial v} = -\frac{u}{v^2}$ $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = v\left(\frac{-u}{v^2}\right) - u\left(\frac{1}{v}\right) = \frac{-u}{v} - \frac{u}{v} = \frac{-2u}{v}$
13	**If $x = r\cos\theta,\ y = r\sin\theta$ *then find* $\frac{\partial(r,\theta)}{\partial(x,y)}$** **(JAN 2018) & (NOV/DEC 2019)**
	Solution: $x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta;\ \frac{\partial x}{\partial \theta} = r(-\sin\theta)$ and $y = r\sin\theta \Rightarrow \frac{\partial y}{\partial r} = \sin\theta;\ \frac{\partial y}{\partial \theta} = r\cos\theta$ $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$ $\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(r,\theta)}} = \frac{1}{r}$
14	**Find Taylor's series expansion of x^y near the point (1,1) up to first degree terms.**
	Solution: The Taylor's series expansion is given by $f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \ldots$ <table><tr><td>$f(x,y) = x^y$</td><td>$f(1,1) = 1$</td></tr><tr><td>$f_x(x,y) = yx^{y-1}$</td><td>$f_x(1,1) = 1$</td></tr><tr><td>$f_y(x,y) = x^y \log x$</td><td>$f_y(1,1) = 0$</td></tr></table> $f(x,y) = 1 + (x-1)(1) + (y-1)(0)$ $= 1 + x - 1$ $x^y = x$
15	**Obtain Taylor's series expansion of e^{x+y} in powers of x and y up to first degree terms.**
	Solution: The Taylor's series expansion is given by $f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \ldots$ <table><tr><td>$f(x,y) = e^{x+y}$</td><td>$f(0,0) = 1$</td></tr><tr><td>$f_x(x,y) = e^{x+y}$</td><td>$f_x(0,0) = 1$</td></tr><tr><td>$f_y(x,y) = e^{x+y}$</td><td>$f_y(0,0) = 1$</td></tr></table> $f(x,y) = f(0,0) + [(x-0) f_x(0,0) + (y-0) f_y(0,0)]$

	$=1+x(1)+y(1)$ $e^{x+y}=1+x+y$
16	**State the conditions for maxima and minima of $f(x,y)$.**
	Solution: If $f_x(a,b)=0,\ f_y(a,b)=0$ and $f_{xx}(a,b)=A,\ f_{xy}(a,b)=B,\ f_{yy}(a,b)=C$ then (i) $f(x,y)$ has maximum value at (a,b) if $AC-B^2>0$ and A< 0 or C< 0 (ii) $f(x,y)$ has minimum value at (a,b) if $AC-B^2>0$ and A > 0 or C> 0
17	**Find the maxima and minima of $f(x,y)=x^2+y^2+6x+4y+12$.**
	Solution: Given $f(x,y)=x^2+y^2+6x+4y+12$ $f_x=2x+6=0\Rightarrow x=-3;\quad f_y=2y+4=0\Rightarrow y=-2.$ The stationary point is (-3, -2). $A=f_{xx}=2;\ B=f_{xy}=0;\ C=f_{yy}=2,$ $AC-B^2=4>0$ and $A>0$. $\therefore f$ is minimum at (−3, -2) and the minimum value is $f(-3,-2)=(-3)^2+(-2)^2+6(-3)+4(-2)+12=25-26=-1.$
18	**Find the possible extreme point of $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$.**
	Solution: $f(x,y)=x^2+y^2+\frac{2}{x}+\frac{2}{y}$ $\frac{\partial f}{\partial x}=2x-\frac{2}{x^2};$ $\frac{\partial f}{\partial x}=0\Rightarrow 2x-\frac{2}{x^2}=0;\ 2x=\frac{2}{x^2};\ x^3=1\ \Rightarrow x=1$ $\frac{\partial f}{\partial y}=2y-\frac{2}{y^2};$ $\frac{\partial f}{\partial y}=0\Rightarrow 2y-\frac{2}{y^2}=0;\ 2y=\frac{2}{y^2};\ y^3=1\ \Rightarrow y=1$ $\therefore$ The possible extreme point is $(1,1)$.
19	**A rectangular box open at the top is to have a maximum capacity whose surface area is 648 square centimeters. Formulate the maximization function to find the dimensions of the box.**
	Solution: Let x,y,z be the length,breadth and height of the box. volume $=length\times breadth\times height=xyz$ (*Volume to be maximized*) *Let* $f(x,y,z)=xyz$ Since the rectangular box is of open at the top, Surface area on the top is zero. $\therefore$ Total surface area of the box $=xy+2yz+2zx=648$ (*given*) *Let* $g(x,y,z)=xy+2yz+2zx-648$ The optimization function to find the dimensions of the box is $F(x,y,z)=f(x,y,z)+\lambda g(x,y,z)$ where λ is langrange multiplier.

	$Hence,\ F(x,y,z) = xyz + \lambda\left(xy + 2yz + 2zx - 648\right)$
20	**Formulate the optimization function to find the volume of the largest rectangular solid which can be inscribed in an ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.
	Solution: Let the sides of the rectangular box be $2x,\ 2y,\ 2z$. The largest rectangular parallelepiped inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ will have its corners on the ellipsoid and its sides parallel to the coordinate plane. Hence its corners are $(\pm x, \pm y, \pm z)$. $\text{Volume}\quad V = 2x \times 2y \times 2z$ $= 8xyz$ $\text{Let}\quad f = 8xyz$ $g = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ The optimization function to find the dimensions of the box is $F(x,y,z) = f(x,y,z) + \lambda g(x,y,z) \quad (i.e.) \quad F = f + \lambda g$ $F(x,y,z) = (8xyz) + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$ where λ is langrange multiplier.
	PART – B
1	**For the given function** $z = \tan^{-1}\left(\frac{x}{y}\right) - (xy)$**, verify whether the statement** $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ **is correct or not.** **(NOV/DEC 2019)**
2	**If** $u = \left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$**, then find the value of** $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$. **(JAN 2018)**
3	**If** $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$**, then find** $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}$. **(JAN 2019)**
4	**If** $u = (x-y)\, f\left(\frac{y}{x}\right)$**, then find** $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x \partial y}+y^2\frac{\partial^2 u}{\partial y^2}$
5	**If** $u = f\left(2x-3y, 3y-4z, 4z-2x\right)$**, then find** $\frac{1}{2}\frac{\partial u}{\partial x}+\frac{1}{3}\frac{\partial u}{\partial y}+\frac{1}{4}\frac{\partial u}{\partial z}$. **(JAN 2019)**
6	**Find** $\frac{du}{dt}$**, if** $u = xy+yz+zx$ **where** $x = t,\ y = e^t,\ z = t^2$.
7	**Find** $\frac{du}{dt}$**, if** $u = x-y+z$ **where** $x = e^{2t},\ y = e^{2t}\cos 3t,\ z = e^{2t}\sin 3t$

8	**If $z = f(x,y)$, where $x = u^2 - v^2$, $y = 2uv$, then prove that $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$.**
9	**If $z = f(x,y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$ then show that $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$**
10	**If $u = 2xy, v = x^2 - y^2$ and $x = \cos\theta, y = \sin\theta$, find $\frac{\partial(u,v)}{\partial(r,\theta)}$**
11	**Find the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1},\ y_2 = \frac{x_1 x_3}{x_2},\ y_3 = \frac{x_1 x_2}{x_3}$.**
12	**If $u = x+y+z,\quad uv = y+z,\quad uvw = z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.**
13	**If $u = \frac{1}{x},\quad v = \frac{x^2}{y},\quad w = x+y+zy^2$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.**
14	**If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$ and $w = x + y + z$, prove that they are functionally dependent and also determine the functional relationship between u, v, w**
15	**Let $u = 3x + 2y - z, v = x - 2y + z$ and $w = x(x + 2y - z)$. Are u, v, w functionally related? If so find the relation? (NOV/ DEC 2020)**
16	**Obtain the Taylor's series expansion of $e^x \sin y$ in powers of x and y up to third degree terms. (NOV/DEC 2019)**
17	**Expand $e^x \log(1+y)$ in powers of x and y up to third degree using Taylor's series.**
18	**Find the Taylor's series expansion of $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$ in powers of (x+2) and (y-1) up to second degree term (NOV/ DEC 2020)**
19	**Expand the function $\sin(xy)$ in powers of $(x-1)$ *and* $\left(y - \frac{\pi}{2}\right)$ as a Taylor series.**
20	**Find the maximum and minimum values of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$**
21	**Find the maximum and minimum values of $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$. (APR/MAY 2019)**
22	**A thin closed rectangular box is to have one edge equal to twice the other and constant volume 72m³. Find the least surface area of the box. (NOV/DEC 2019)**
23	**Find the dimension of the rectangular box open at the top of maximum capacity 432 cc. (NOV/DEC 2020)**
24	**Find the length of the shortest line from the point $\left(0,0,\frac{25}{9}\right)$ to the surface $z = xy$.**
25	**Find the shortest and longest distances from the point (1,2,-1) to the sphere $x^2 + y^2 + z^2 = 24$. (APR/MAY 2019)**
UNIT IV – INTEGRAL CALCULUS	
PART – A	

1	**State the properties of definite integrals.**
	Solution: (i) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \end{cases}$ (ii) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$
2	**Set up an expression for $\int_1^3 e^x dx$ as a limit of sums**
	Solution: $f(x)=e^x,\ a=1,\ b=3,\ \Delta x=\frac{b-a}{n}=\frac{2}{n},\ x_i=a+i\Delta x$ (By Riemann Sum method) $\int_1^3 e^x dx=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{2i}{n}\right)\frac{2}{n}=\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n} e^{\left(1+\frac{2i}{n}\right)}$
3	**Evaluate the following integrals using fundamental theorem of calculus $\int_1^8 \sqrt[3]{x}\,dx$**
	Solution: Given: $f(x)=x^{1/3},\ a=1, b=8$ Antiderivative: $F(x)=\frac{x^{4/3}}{4/3}$ By using Fundamental theorem of calculus $\int_1^8 \sqrt[3]{x}\,dx=F(b)-F(a)=F(8)-F(1)$ $=\frac{8^{4/3}}{4/3}-\frac{1}{4/3}=\frac{3}{4}\left(8^{1/3}\right)^4-\frac{3}{4}=\frac{3}{4}2^4-\frac{3}{4}=\frac{3}{4}[16-1]=\frac{45}{4}$
4	**Evaluate $\int\left(10x^4-2\sec^2 x\right)dx$**
	Solution: $\int\left(10x^4-2\sec^2 x\right)dx=10\int x^4dx-2\int \sec^2 x\,dx$ $=10\frac{x^5}{5}-2\tan x+c$ $=5x^5-2\tan x+c$
5	**Evaluate $\int_1^2\left(-3x^{1/2}+\frac{1}{x^2}\right)dx$.**
	Solution: $\int_1^2\left(-3x^{1/2}+\frac{1}{x^2}\right)dx=-3\int_1^2 x^{1/2}\,dx+\int_1^2\frac{1}{x^2}\,dx$ $=-3\left[\frac{x^{3/2}}{3/2}\right]_1^2+\left[-\frac{1}{x}\right]_1^2$ $=\left[-2x^{3/2}-\frac{1}{x}\right]_1^2$

	$=\left[\left(-2\left(2^{3/2}\right)-\frac{1}{2}\right)-\left(-2\left(1^{3/2}\right)-1\right)\right]$ $=\left[-2\left(2\sqrt{2}\right)-\frac{1}{2}+3\right]=-4\sqrt{2}+\frac{5}{2}$
6	**Evaluate by substitution method** $\int\left(2x\sqrt{1+x^2}\right)dx$
	Solution: Let $u=1+x^2, du=2xdx$ $\int\left(2x\sqrt{1+x^2}\right)dx=\int\sqrt{u}du=\frac{u^{3/2}}{3/2}+c$ $=\frac{2}{3}u^{3/2}+c$ $=\frac{2}{3}\left(1+x^2\right)^{3/2}+c$
7	**Find** $\int\left(\frac{\cos\theta}{\sqrt{\sin\theta}}\right)d\theta$ **by substitution method.**
	Solution: Let $I=\int\left(\frac{\cos\theta}{\sqrt{\sin\theta}}\right)d\theta$ Put $u=\sin\theta$ then $du=\cos\theta\, d\theta$ $\therefore I=\int\frac{du}{\sqrt{u}}=\int(u)^{-1/2}du=\frac{u^{(-1/2)+1}}{\left(\frac{-1}{2}\right)+1}=2\sqrt{u}+c=2\sqrt{\sin\theta}+c$
8	**If f is continuous and $\int_0^4 f(x)dx=10$, then find $\int_0^2 f(2x)dx$.** **(NOV/DEC 2018)**
	Solution: Given f is continuous, let $2x=t$, differentiating $2dx=dt$ or $dx=\frac{dt}{2}$. when $x=2$, then $t=4$, when $x=0$, then $t=0$. $\therefore\int_0^2 f(2x)dx=\int_0^4 f(t)\frac{dt}{2}=\frac{1}{2}\int_0^4 f(t)dt=\frac{10}{2}=5$
9	**Evaluate** $\int\tan^{-1}x\, dx$.
	Solution: $u=\tan^{-1}x$; $dv=dx$ $du=\frac{dx}{1+x^2}$; $v=\int dv=\int dx=x$ $\left(\because \int udv=uv-\int vdu\right)$ $\int\tan^{-1}x\,dx=uv-\int vdu$ $=x\tan^{-1}x-\int x\frac{dx}{1+x^2}$

	$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx \qquad \left(\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right)$ $= x\tan^{-1}x - \frac{1}{2}\log\left(1+x^2\right)+c$
10	**Find the Integral of $x\sin x$ using integration by parts.**
	Solution: $u = x$, $dv = \sin x dx$ $du = dx$, $v = \int dv = \int \sin x\, dx = -\cos x$ $\left(\because \int u dv = uv - \int v du\right)$ $\int x\sin x\; dx = x(-\cos x) - \int(-\cos x)dx$ $= -x\cos x + \sin x + c$ $= \sin x - x\cos x + c$
11	**Evaluate $\int_0^{\pi/2} \sin^6 x\; \cos^5 x\, dx$.**
	Solution: $\int_0^{\pi/2} \sin^m x\cos^n x\, dx = \frac{n-1}{m+n}\frac{n-3}{m+n-2}\frac{n-5}{m+n-4}\cdots\frac{1}{m+1},$ when m is an even and n is an odd integer, here $m=6, n=5$ $= \frac{4}{11}\frac{2}{9}\frac{1}{7} = \frac{8}{693}$
12	**Use the hyperbolic substitution to show that $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)+c$**
	Solution: Let $I = \int \frac{dx}{\sqrt{x^2+a^2}}$ Put $x = a\sinh t$ then $dx = a\cosh t dt$ We know that $\cosh^2 t - \sinh^2 t = 1 \Rightarrow \cosh^2 t = 1+\sinh^2 t$ $I = \int \frac{a\cosh t dt}{\sqrt{a^2\sinh^2 t + a^2}} = \int \frac{a\cosh t dt}{a\sqrt{\sinh^2 t+1}} = \int \frac{a\cosh t dt}{a\sqrt{\cosh^2 t}} = \int \frac{a\cosh t dt}{a\cosh t} = \int dt = t + c$ $x = a\sinh t \Rightarrow \sinh t = \frac{x}{a},\; t = \sinh^{-1}\left(\frac{x}{a}\right)$ $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{x}{a}\right)+c$
13	**Evaluate $\int \frac{x-9}{(x+5)(x-2)}dx$**
	Solution: Let $I = \int \frac{x-9}{(x+5)(x-2)}dx$ $\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$ $x-9 = A(x-2) + B(x+5)$ When $x = 2 \Rightarrow -7 = B(7) \Rightarrow B = -1$

	When $x=-5 \Rightarrow -14 = A(-7) \Rightarrow A=2$ $I = \int \frac{2}{x+5}dx + \int \frac{1}{x-2}dx$ $= 2\int \frac{1}{x+5}dx + \int \frac{1}{x-2}dx$ $= 2\ln\lvert x+5\rvert + \ln\lvert x-2\rvert + c$
14	**Evaluate** $\int\left(\frac{x}{x-6}\right)dx$
	Solution: Let $I = \int\left(\frac{x}{x-6}\right)dx = \int\left(\frac{x-6+6}{x-6}\right)dx = \int\left(\frac{x-6}{x-6}\right)dx + \int\left(\frac{6}{x-6}\right)dx$ $= \int dx + 6\int\left(\frac{1}{x-6}\right)dx$ $= x + 6\ln(x-6) + c$
15	**Determine whether integral** $\int_1^\infty \frac{\ln x}{x}dx$ **is convergent or divergent. Evaluate it, if it is convergent.** **(NOV/DEC 2020)**
	Solution: $\int_1^\infty \frac{\ln(x)}{x}dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x}dx$ $= \lim_{t\to\infty}\int_1^t \ln x\, d(\log x) = \lim_{t\to\infty}\left(\frac{(\log x)^2}{2}\right)_1^t = \frac{1}{2}\lim_{t\to\infty}\left((\log t)^2 - (\log 1)^2\right) = \frac{1}{2}\lim_{t\to\infty}\left((\log t)^2\right) = \infty$ Hence it is divergent
16	**Evaluate** $\int_3^\infty \frac{dx}{(x-2)^{3/2}}$ **and determine whether it is convergent or divergent.**
	Solution: Let $I = \int_3^\infty \frac{dx}{(x-2)^{3/2}} = \lim_{b\to\infty}\int_3^b \frac{dx}{(x-2)^{3/2}} = \lim_{b\to\infty}\int_3^b (x-2)^{-3/2}dx$ $= \lim_{b\to\infty}\left(\frac{(x-2)^{-3/2+1}}{\frac{-3}{2}+1}\right)_3^b = \lim_{b\to\infty}\left(\frac{(x-2)^{-1/2}}{\frac{-1}{2}}\right)_3^b = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{x-2}}\right)_3^b$ $= \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b-2}} - \frac{-2}{\sqrt{3-2}}\right) = 2$ Hence the given integral is convergent.
17	**An aquarium 5 m long, 10 m wide and 3 m deep is filled with seawater of density 1030 kg/m³ to a depth of 2.5 m, then find the hydrostatic pressure and force at the bottom of the pool.**
	Solution: The hydrostatic pressure at the bottom of the pool is: Pressure = density × gravity × depth = 1030 × 9.8 × 2.5 = 25,235 The hydrostatic force at the bottom of the pool is: Force = pressure × area

$= 1030 \times 9.8 \times 2.5 \times 5 \times 10 = 1{,}261{,}750$

18 **A rod has a length of 40 cm. The rod's density changes linearly along its length from 20 g/cm to 60 g/cm. Find the center of mass of the rod.**

Let's first derive the density distribution function.

The equation of straight line passing through the points $\rho(0)=20,\ \rho(40)=60$ is given by

$\dfrac{\rho-\rho(0)}{\rho(40)-\rho(0)}=\dfrac{x-0}{40-0}\Rightarrow\dfrac{\rho-20}{60-20}=\dfrac{x}{40}\Rightarrow\rho=x+20,$ where ρ is measured in g/cm and x is measured in cm.

Calculate the mass m and the first moment M_0 of the rod.

$$m=\int_a^b\rho(x)\,dx=\int_0^{40}(x+20)\,dx=\left[\frac{x^2}{2}+20x\right]_0^{40}=800+800=1600$$

$$M_0=\int_a^b x\rho(x)\,dx=\int_0^{40}x(x+20)\,dx=\int_0^{40}\left(x^2+20x\right)dx=\left[\frac{x^3}{3}+10x^2\right]_0^{40}=\frac{64000}{3}+16000\approx 37333g/cm$$

Hence, the center of mass $G(\bar{x})$ is located at the point

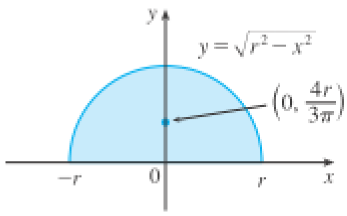
$$\bar{x}=\frac{M_0}{m}=\frac{37333}{1600}\approx 23.3cm$$

19 **Find the center of mass of a semicircular plate of radius r.**

Solution:

Let $f(x)=\sqrt{r^2-x^2}$ and $a=-r,\ b=r$.

Here there is no need to use the formula to calculate $\bar{x}$ because, by the symmetry principle, the center of mass lie on the y-axis so $\bar{x}=0$.



The area of the semicircle is $A=\frac{1}{2}\pi r^2$, so

$$\bar{y}=\frac{1}{A}\int_{-r}^{r}\frac{1}{2}\left[f(x)\right]^2dx=\frac{1}{\frac{1}{2}\pi r^2}\int_{-r}^{r}\frac{1}{2}\left[\sqrt{r^2-x^2}\right]^2dx$$

$$=\frac{1}{\pi r^2}\int_{-r}^{r}\left(r^2-x^2\right)dx$$

$$=\frac{2}{\pi r^2}\int_0^{r}\left(r^2-x^2\right)dx$$

$$=\frac{2}{\pi r^2}\left(xr^2-\frac{x^3}{3}\right)_0^r=\frac{2}{\pi r^2}\left(r^3-\frac{r^3}{3}\right)=\frac{2}{\pi r^2}\left(\frac{2r^3}{3}\right)=\frac{4r}{3\pi}$$

The center of mass is located at the point $\left(0,\dfrac{4r}{3\pi}\right)$

20 **Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points (-1,1), (2,-1) and (3,2) respectively.**

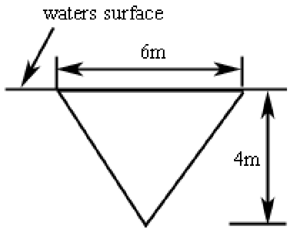
Solution:

Moment of the system about the y-axis is $M_y=\sum_{i=1}^{n}m_ix_i=3(-1)+4(2)+8(3)=29$

Moment of the system about the x-axis is $M_x=\sum_{i=1}^{n}m_iy_i=3(1)+4(-1)+8(2)=15$

The coordinates $(\bar{x},\bar{y})$ of the center of mass are given in terms of the moments

	$\bar{x}=\frac{M_y}{m}$ and $\bar{y}=\frac{M_x}{m}$ where $m=\sum m_i=3+4+8=15$ $\bar{x}=\frac{29}{15}$ and $\bar{y}=\frac{15}{15}=1$ The center of mass is $\left(\frac{29}{15},1\right)$
PART – B	
1	**Evaluate $\int_0^3\left(x^3-6x\right)dx$ by using Riemann sum with n sub intervals. (NOV/DEC 2019)**
2	**Evaluate the following integrals by using Riemann sum by taking right end points as sample points. Hence verify it by using fundamental theorem of calculus $\int_1^4\left(x^2+2x-5\right)dx$.**
3	**Evaluate $\int e^x \sin x\, dx$ by using integration by parts. (NOV/DEC 2019)**
4	**Evaluate $\int \frac{xe^{2x}}{(1+2x)^2}dx$ by using integration by parts.**
5	**Prove the reduction formula $\int_0^{\pi/2}\sin^n x dx=\frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x dx$. Hence by using it evaluate $\int_0^{\pi/2}\sin^7 x dx$.**
6	**Prove that $\int_{-\pi}^{\pi}\sin mx \sin nx dx=\begin{cases}0 & m\neq n\\ \pi & m=n\end{cases}$ where m and n are positive integrals. (NOV/DEC 2020)**
7	**Evaluate $\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x+3\cos x+2}dx$ (APR/MAY 2019)**
8	**Evaluate $\int_0^{\pi/2}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$.**
9	**Evaluate $\int_{\sqrt{2}/3}^{2/3}\frac{dx}{x^5\sqrt{9x^2-1}}$. (NOV/DEC 2018)**
10	**Evaluate the integral 1) $\int\left(x^3\sqrt{x^2+1}\right)dx$ and 2) $\int_0^1\frac{1}{\left(1+\sqrt{x}\right)^4}dx$ (NOV/DEC 2020)**
11	**Evaluate the integral $\int\frac{x^2-2x-1}{(x-1)^2\left(x^2+1\right)}dx$ (NOV/DEC 2020)**
12	**Evaluate $\int\frac{2x+1}{x^3+2x^2-x-2}dx$ by partial fraction method.**
13	**Evaluate $\int\frac{2x+5}{\sqrt{x^2-2x+10}}dx$. (APR/MAY 2019)**
14	**Evaluate $\int\sqrt{a^2-x^2}dx$ by using substitution rule. (NOV/DEC 2019)**

15	**Find the value of p for which the integral $\int_0^1 x^p \ln(x)\,dx$ converges and evaluate the integral for those value of p.** **(NOV/DEC 2020)**
16	**Evaluate $\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx$ and hence discuss their convergence.**
17	**Find the hydrostatic force on a circular plate of radius 2 that is submerged 6 meters in the water.**
18	**Determine the hydrostatic force on the following triangular plate that is submerged in water as shown.** 
19	**Determine the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.**
20	**Determine the center of mass for the region bounded by $y = 2\sin(2x)$, $y = 0$ on the interval $\left[0, \frac{\pi}{2}\right]$.**
	UNIT V – MULTIPLE INTEGRALS
	PART- A
1	**Evaluate $\int_0^2 \int_0^1 y\,dy\,dx$**
	Solution: $I = \int_0^2 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\int_0^2 (1^2 - 0)dx = \frac{1}{2}\int_0^2 dx = \frac{1}{2}[x]_0^2 = \frac{1}{2}(2-0) = 1$
2	**Evaluate $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$.** **(APR/MAY 2017)**
	Solution: $I = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi}\cos^2\theta\,d\theta$ $= \frac{1}{2}\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta \quad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$ $= \frac{1}{4}\int_0^{\pi}(1+\cos 2\theta)\,d\theta$ $= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = \frac{1}{4}(\pi) = \frac{\pi}{4}$
3	**Evaluate $\int_1^2 \int_0^{x^2} x\,dy\,dx$** **(NOV/DEC 2019)**
	Solution:

$$\int_1^2\int_0^{x^2} x\,dydx = \int_1^2 x\left(\int_0^{x^2} dy\right)dx = \int_1^2 x\left(y\right)_0^{x^2} dx = \int_1^2 x\left(x^2\right)dx = \int_1^2 x^3dx = \left(\frac{x^4}{4}\right)_1^2 = \left(\frac{2^4}{4}-\frac{1^4}{4}\right) = \left(\frac{16}{4}-\frac{1}{4}\right) = \frac{15}{4}$$

4 **Change the order of integration in $\int_0^1\int_{y^2}^{y} f(x,y)\,dxdy$.** **(NOV/DEC 2018)**

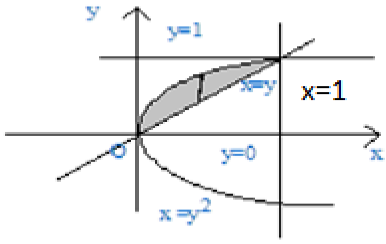
Solution:

Given $I = \int_{y=0}^{y=1}\int_{x=y^2}^{x=y} f(x,y)\,dx\,dy$

Limit of $x: y^2$ to y and Limit of $y: 0$ to 1

After changing the order of integration

$I = \int_{x=0}^{x=1}\int_{y=x}^{y=\sqrt{x}} f(x,y)\,dydx$



5 **Check the correct form of $\int_0^1\int_0^{x^2}(1+x)dxdy$**

Solution:

Given $\int_0^1\int_0^{x^2}(1+x)dxdy$

The limits for the inner integral are functions of x, then the first integral is with respect to y

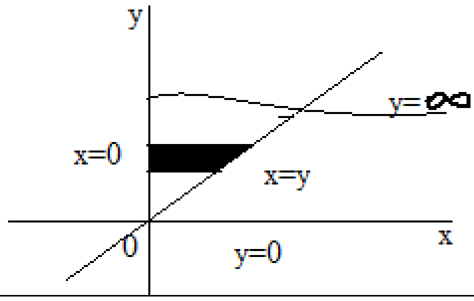
$\therefore \int_0^1\int_0^{x^2}(1+x)dxdy = \int_0^1\int_0^{x^2}(1+x)dydx$ [correct form]

6 **Transform the integration $\int_0^\infty\int_0^y dx\,dy$ into polar co-ordinates.**

Solution:

Let $x = r\cos\theta$ and $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\int_0^\infty\int_0^y dx\,dy = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=0}^{\infty} r\,dr\,d\theta$$



7 **Shade the region of integration in $\int_0^a\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$.**

Solution:

The given order of the integration is not correct.

So, rewrite the order

$$\int_{x=0}^{x=a}\int_{y=\sqrt{ax-x^2}}^{y=\sqrt{a^2-x^2}} dydx.$$

$$y = \sqrt{ax-x^2} \Rightarrow x^2+y^2-ax = 0$$

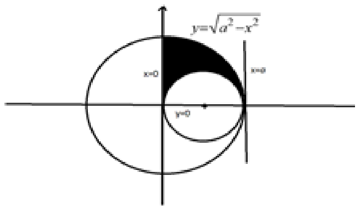
$$x^2-ax+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+y^2 = 0$$

$$\left(x-\frac{a}{2}\right)^2+y^2 = \left(\frac{a}{2}\right)^2$$

which is a circle with center at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$.

$$y=\sqrt{a^2-x^2}\Rightarrow y^2=a^2-x^2\ \Rightarrow x^2+y^2=a^2$$

which is a circle with center at (0,0) and radius *a.*



8 **Find the area bounded by the lines $x=0, y=1, y=x$ using double integration.**

Solution:

$$\text{Area}=\iint_R dxdy=\int_0^1\int_0^y dxdy=\int_0^1[x]_0^y\,dy=\int_0^1 ydy=\left[\frac{y^2}{2}\right]_0^1=\frac{1}{2}.$$

9 **Evaluate $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$.**

Solution:

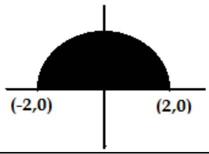
$$I=\int_0^1\int_0^1\int_0^1 e^{x+y+z}dzdydx=\int_0^1\int_0^1\int_0^1 e^xe^ye^zdzdydx$$

$$=\left[\int_0^1 e^xdx\right]\left[\int_0^1 e^ydx\right]\left[\int_0^1 e^zdx\right]$$

$$=\left[e^x\right]_0^1\ \left[e^x\right]_0^1\ \left[e^x\right]_0^1\ =\left(e^1-e^0\right)\left(e^1-e^0\right)\left(e^1-e^0\right)$$

$$=(e-1)^3$$

10 **Evaluate $\iint_R$ dxdy, where R is the shaded region in the figure.**



Solution:

$\iint_R dxdy=$ Area of the shaded region

The shaded region is the semicircle with radius 2.

Area of the shaded portion = $\frac{\pi r^2}{2}=\frac{\pi 2^2}{2}=2\pi$ square units

11 **Evaluate $\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dx\,dy\,dz$.**

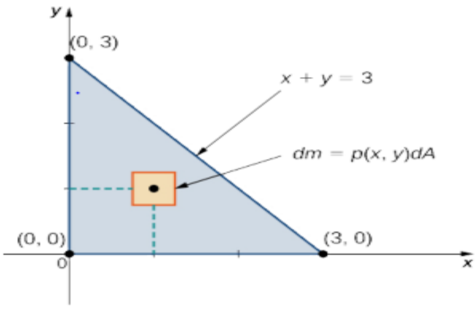
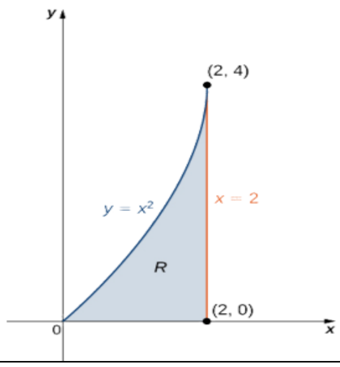
Solution:

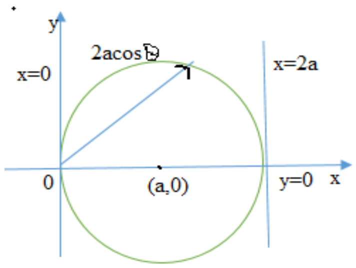
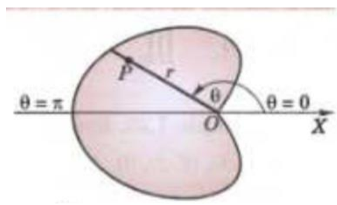
$$I=\int_0^4\int_0^x\int_0^{\sqrt{x+y}} z\,dz\,dy\,dx$$

$$=\int_0^4\int_0^x\left[\frac{z^2}{2}\right]_0^{\sqrt{x+y}}dy\,dx$$

$$=\frac{1}{2}\int_0^4\int_0^x(x+y)\,dy\,dx$$

$$=\frac{1}{2}\int_0^4\left[xy+\frac{y^2}{2}\right]_0^x dx$$

	$=\frac{1}{2}\int_0^4\left(x^2+\frac{x^2}{2}\right)dx$ $=\frac{3}{4}\int_0^4 x^2dx$ $=\frac{3}{4}\left[\frac{x^3}{3}\right]_0^4=16$
12	**Express the region $x\geq 0, y\geq 0, z\geq 0, x^2+y^2+z^2\leq 1$ by triple integration. (NOV/DEC 2019)**
	Solution: Given region is the positive octant of the sphere. x varies from 0 to 1 y varies from 0 to $\sqrt{1-x^2}$ z varies from 0 to $\sqrt{1-x^2-y^2}$ $\therefore I=\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} dzdydx$
13	**Consider a triangular lamina R with vertices (0,0),(0,3),(3,0) and with density $\rho(x,y)=xy\,kg/m^2$. Find the moments M_x and M_y**
	Solution: The moments are $M_x=\iint_R y\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2dydx=\frac{81}{20}$ $M_y=\iint_R x\rho(x,y)dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2ydydx=\frac{81}{20}$ 
14	**Find the mass of the lamina of density $\rho(x,y)=x+y$ occupying the region R under the curve $y=x^2$ in the interval $0\leq x\leq 2$**
	Solution: We compute the mass m $m=\iint_R dm=\iint_R \rho(x,y)dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x^2}(x+y)dydx$ $=\int_{x=0}^{x=2}\left[xy+\frac{y^2}{2}\right]_{y=0}^{y=x^2}dx=\int_{x=0}^{x=2}\left[x^3+\frac{x^4}{2}\right]dx$ $=\left[\frac{x^4}{4}+\frac{x^5}{10}\right]_{x=0}^{x=2}=\frac{36}{5}$ 
15	**Find the moment of inertia of a lamina covering the inside of the unit circle, with density function $\rho(x,y)=1-x^2-y^2$**
	Solution: The moment of inertia of the entire lamina is $I=\iint_D (x^2+y^2)\rho(x,y)dA$ By polar co-ordinates, $r^2=x^2+y^2$

	$I=\iint_D r^2\rho(x,y)dA=\int_0^{2\pi}\int_0^1 r^2(1-r^2)rdrd\theta$ $=\int_0^{2\pi}\int_0^1(r^3-r^5)drd\theta$ $=\int_0^{2\pi}\frac{d\theta}{12}=\frac{\pi}{6}$
16	**Change the order of integration in $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}\left(x^2+y^2\right)dx\,dy$.**
	Solution: $I=\int_{x=-a}^{x=a}\int_{y=0}^{y=\sqrt{a^2-x^2}}\left(x^2+y^2\right)dy\,dx$ (correct form) Given $y=0$ and $y=\sqrt{a^2-x^2}\Rightarrow y^2=a^2-x^2\Rightarrow x^2+y^2=a^2$ $x=-a$ and $x=a$ By changing the order of Integration $I=\int_{y=0}^{y=a}\int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}}\left(x^2+y^2\right)dx\,dy$ 
17	**Transform the integration $\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx$ to polar co-ordinates.**
	Solution: Let $x=r\cos\theta$ and $y=r\sin\theta$, $dx\,dy=r\,dr\,d\theta$ $x^2+y^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2$ Here $y=\sqrt{2ax-x^2}\Rightarrow y^2=2ax-x^2$ $\Longrightarrow(rsin\theta)^2=2arcos\theta-r^2cos^2\theta$ $\Longrightarrow r^2sin^2\theta+r^2cos^2\theta=2arcos\theta\ \Rightarrow r^2=2arcos\theta$ $\Longrightarrow r=2acos\theta$ $\therefore\quad r:0\to 2acos\theta$ and $\theta:0\to\frac{\pi}{2}$ $\int_{x=0}^{2a}\int_{y=0}^{\sqrt{2ax-x^2}}\left(x^2+y^2\right)dy\,dx=\int_0^{\pi/2}\int_0^{2a\cos\theta}r^3dr\,d\theta$
18	**Calculate by double integration, the volume generated by the revolution of the cardioid $r=a(1-\cos\theta)$ about its axis.**
	Solution: Required Volume $=\int_0^{\pi}\int_0^{a(1-\cos\theta)}2\pi r^2\sin\theta\,dr\,d\theta$ $=2\pi\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^{a(1-\cos\theta)}\sin\theta\,d\theta$ $=\frac{2\pi a^3}{3}\int_0^{\pi}(1-\cos\theta)^3\sin\theta\,d\theta=\frac{2\pi a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi}=\frac{8\pi a^3}{3}$ 
19	**Change the order of integration in $\int_0^1\int_y^{2-y}xy\,dx\,dy$.**
	Solution:

Given, $I = \int_{y=0}^{y=1}\int_{x=y}^{x=2-y} xy\ dx\,dy$

x limits: y to $2-y$; y limits: 0 to 1

i.e., $x = y$ and $x = 2-y \Rightarrow x+y=2$

After changing order of integration

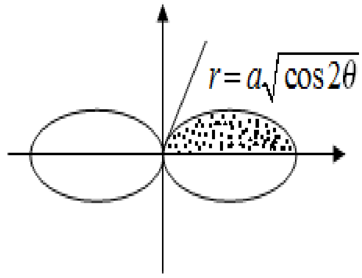
$$I = \int_0^1\int_0^x xy\,dy\,dx + \int_1^2\int_0^{2-x} xy\,dy\,dx$$

20 **Compute the entire area bounded by $r^2 = a^2\cos 2\theta$.**

Solution:

Given $r^2 = a^2\cos 2\theta \Rightarrow r = a\sqrt{\cos 2\theta}$

Area $= 4\times$ Area in first quadrant



$$A = \iint_R r\,dr\,d\theta = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/4}\left[\frac{r^2}{2}\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 4\int_{\theta=0}^{\pi/4}\left[\frac{a^2\cos 2\theta}{2}\right]d\theta = \frac{4}{2}a^2\int_{\theta=0}^{\pi/4}\cos 2\theta\,d\theta$$

$$= 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = 2a^2\left[\frac{\sin 2(\pi/4)}{2} - 0\right]$$

$$= a^2\sin\frac{\pi}{2} = a^2$$

PART- B

1	**Evaluate $\iint xydxdy$ over the positive quadrant of the circle $x^2+y^2=a^2$. (NOV/DEC 2019)**
2	**Evaluate $\iint xydxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$**
3	**Evaluate $\iint_R x^2dxdy$ where R is the region in the first quadrant bounded by the lines $x=y, y=0, x=8$ and the curve $xy=16$.**
4	**Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dxdy$. (NOV/DEC 2016)**
5	**Find the area bounded by $y^2=4x$ and $x^2=4y$ by using double integrals. (NOV/DEC 2019)**
6	**Find the area bounded by the parabolas $y^2=4-x$ and $y^2=x$ (NOV/DEC 2018)**
7	**Find the area of the cardioid $r=a(1+\cos\theta)$.**
8	**Find the area included between the curve $r=a(\sec\theta+\cos\theta)$ and its asymptote**
9	**Change the order of integration in $\int_0^1\int_y^{2-y}$ xydxdy and evaluate it. (NOV/DEC 2020)**
10	**Change the order of integration in $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}dy\,dx$ and hence evaluate it. (APR/MAY 2019)**

11	**Transform the integral into polar co-ordinates and hence evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$ and hence find the value of $\int_0^\infty e^{-x^2} dx$.**
12	**Evaluate $\iint_D xy\sqrt{1-x-y}\,dxdy$ where D is the region bounded by x = 0,y = 0, x + y = 1 using the transformation x + y = u,y = uv (NOV/DEC 2020)**
13	**Evaluate by changing into polar coordinates $\int_0^a \int_y^a \frac{x^2 dxdy}{\sqrt{x^2+y^2}}$. (APR/MAY 2019)**
14	**Evaluate $\iint_R r^2 \sin\theta\, drd\theta$ where R is the semi circle $r = 2a\cos\theta$ above the initial line**
15	**Evaluate $\iiint_V xyz\, dxdydz$ where V is the volume of the positive octant of the sphere $x^2+y^2+z^2=1$ by transforming into spherical polar coordinates (NOV/DEC 2020)**
16	**Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$**
17	**Evaluate $\int_0^{2a}\int_0^x\int_y^x (xyz)\,dz\,dy\,dx$. (NOV/DEC 2019)**
18	**Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dxdydz$**
19	**Find the volume of the cylinder bounded by $x^2+y^2=4$ and then planes $y+z=4$ and $z=0$ using triple integral (NOV/DEC 2020)**
20	**Evaluate $\iiint_V dx\,dy\,dz$ where V is the finite region of space (tetrahedron) bounded by the planes $x=0, y=0,\ z=0\ \&\ 2x+3y+4z=12$. (N0V/DEC 2018)**
21	**Find the mass and center of mass of a triangular Lamina with vertices (0,0), (1,0) and (0,2) if the density function is $\rho(x,y)=1+3x+y$**
22	**Find the mass and center of mass of a lamina with density function $\rho(x,y)=6x$ covering the triangle D bounded by the x-axis, the line $y=x$ and the line $y=2-x$**
23	**Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disk D with density $\rho(x,y)=\rho$, center the origin, and radius a.**
24	**Find the moment of inertia of the area bounded by the curve $r^2=a^2\cos 2\theta$ about its axis**