SRI VENKATESWARA COLLEGE OF ENGINEERING & TECHNOLOGY, THIRUPACHUR.

MATRICES AND CALCULUS – MA3151

IMPORTANT QUESTIONS IN UNIT WISE

UNIT 1 - MATRICES

PART A

1. Find the sum and product of the eigenvalues of the matrix $\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Solution:

Sum of the Eigen Values = Sum of the diagonal elements

= (-1) + (-1) + (-1) = **-3**

product of the eigenvalues = $\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$

= -1(1-1) -1(-1-1) + (1+1)

= -1(0) -1(-2) + (2)

= 0+2+2

= **4**

2. The product of two eigenvalues of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, find the third eigenvalue

Solution:

Let the eigenvalues of the matrix A be $\lambda_1, \lambda_2, \lambda_3$

Given: $\lambda_1\lambda_2 = 16$

We know that $\lambda_1\lambda_2\lambda_3 = |A|$

$\therefore \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

= 6(9-1) + 2(-6+2) + 2(2-6)

=6(8) + 2(-4) + 2(-4)

=48-8-8

$\boldsymbol{\lambda_1\lambda_2\lambda_3 = 32}$

$16\lambda_3 = 32$

$\lambda_3 = 32 \div 16$

$\boldsymbol{\lambda_3 = 2}$

3. Two of the Eigenvalues of A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the eigenvalues of A^{-1}.

Solution:

Sum of the Eigen Values = sum of the main diagonal elements

= 3+5+3 = 11

Let k be the third Eigenvalues

$\therefore 3+9+k = 11$

$9+k = 11$

k= 2

[Property: If eigenvalues of A are $\lambda_1\lambda_2\lambda_3$, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}$]

$\therefore$ **The Eigenvalues of A^{-1} are** $\boldsymbol{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}}$

4. Two Eigenvalues of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each.Find the eigenvalues of A^{-1}.

Solution:

Given: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Let the eigenvalues of the matrix A be $\lambda_{1,2,}\boldsymbol{\lambda}_3$

Given condition is $\boldsymbol{\lambda}_2 = \boldsymbol{\lambda}_3 = 1$

We have,

Sum of the Eigen values = sum of the main diagonal element s

$\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2+\boldsymbol{\lambda}_3 = 2+3+2$

$\boldsymbol{\lambda}_1+1+1 = 7$

$\boldsymbol{\lambda}_1 +2 = 7$

$\boldsymbol{\lambda}_1 = 5$

Hence, the eigenvalues of A are 1, 1, 5

[Property: if $\boldsymbol{\lambda}$ is an eigenvalues of a non-singular matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} i.e., $\boldsymbol{\lambda}\neq0$]

Eigenvalues of A^{-1} are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$ i.e., **Eigenvalues of A^{-1} are 1, 1, $\frac{1}{5}$**

5. Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Solution:

Given: $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$

Clearly given matrix A is a lower triangular matrix.

[Property: The characteristic roots of a triangular matrix are just the diagonal elements of the matrix]

Hence, by property **the Eigenvalues of A are 2, 3, 4.**

PART B

1. Verify Cayley - Hamilton theorem find A^4 and A^{-1} when $\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Ans: $A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$ $A^4 = \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$

2. Use Cayley Hamilton theorem to find the value of the matrix given by

(i) $f(A) = A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

(ii) $g(A) = A^8-5A^7+7A^6-3A^5+8A^4-5A^3+8A^2-2A+I$ if the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Ans: $f(A) = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ $g(A) = \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$

3. Diagonalise the matrix $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and hence find A^4

Ans: $A^4 = \begin{bmatrix} 22536 & -22482 & 11214 \\ -22482 & 22509 & -11268 \\ 11214 & -11268 & 5661 \end{bmatrix}$

4. Reduce the quadratic form $Q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$ into canonical form by an orthogonal transformation

Ans: C.F = $8y_1^2+2y_2^2+2y_3^2$

5. Reduce the quadratic form to canonical form by an orthogonal reduction $2x_1x_2 +2x_1x_3-2x_2x_3$ also discuss its nature

Ans: C.F = $-2y_1^2+y_2^2+y_3^2$

6. Reduce the quadratic form $x_1^2+2x_2^2+x_3^2-2x_1x_2+2x_2x_3$ to the canonical form through an orthogonal transformation and hence, show that it is positive semi definite. Also give a non-zero set of values (x_1,x_2,x_3) which makes this quadratic form zero

Ans: C.F =$0y_1^2+y_2^2+3y_3^2$

7. Find the principal directions and corresponding factors of extension or contraction of an elastic deformation Y = AX with given

$$A = \begin{bmatrix} 7 & \sqrt{6} \\ \sqrt{6} & 2 \end{bmatrix}$$

Ans: 22.2°, 8 ; 112.21°, 1

8. Find the principal directions and corresponding factors of extension or contraction of an elastic deformation Y = AX with given

$$A = \begin{bmatrix} 3/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{bmatrix}$$

Ans: 35.25°, 2 ; 125.26°, $\frac{1}{2}$

UNIT II- DIFFERENTIAL CALCULUS

PART A

1. Use the squeeze theorem, find the value of $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$

Solution:

Given: $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$

[**The Squeeze Theorem:**

If $f(x) \le g(x) \le h(x)$ where x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$]

We know that, $-1 \le \sin(1/x) \le 1 \Rightarrow -x^2 \le x^2 \sin(\frac{1}{x}) \le x^2$

$\lim_{x\to 0}(-x^2) = \lim_{x\to 0}(x^2) = 0$

$\mathbf{\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0}$ by squeeze theorem.

2. Differentiate the function if $y = \frac{x^2+4x+3}{\sqrt{x}}$

Solution:

Given: $y = \frac{x^2+4x+3}{\sqrt{x}}$

$y = x^2(x^{-1/2}) + 4x(x^{-1/2}) + 3(x^{-1/2}) = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$

$y' = \frac{dy}{dx} = \frac{3}{2}x^{1/2} + 4(\frac{1}{2}x^{-1/2}) + 3(\frac{-1}{2}x^{-3/2}) = \mathbf{\frac{3}{2}\sqrt{x} + 2\frac{1}{\sqrt{x}} - \frac{3}{2}x^{-3/2}}$

3. Differentiate the function if $y = 3e^x + \frac{4}{\sqrt[3]{x}}$

Solution:

Given: $y = 3e^x + 4x^{-1/3}$

$y' = \frac{dy}{dx} = 3e^x + 4(\frac{-1}{3})x^{-4/3} = \mathbf{3e^x - \frac{4}{3}x^{-4/3}}$

4. Find $f'(x)$ if $f(x) = (x^3+2x)e^x$

Solution:

Given: $f(x) = (x^3+2x)e^x$

$f'(x) = (x^3+2x)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^3+2x)$

$= (x^3+2x)e^x + e^x(3x^2+2)$

$= e^x(x^3+2x+3x^2+2) = \mathbf{e^x(x^3+3x^2+2x+2)}$

5. Find the derivatives of the function if y=cosec x+ e^xcot x

Solution:

Given : y=cosecx +e^xcot x

$y' = \frac{dy}{dx}$ = -cosec x cot x + e^x [-$cosec^2$x]+cot x [e^x]

= -cosec x cot x -$e^x cosec^2$x + e^xcot x.

6.Find f'(x) if f(x) = x e^xcosec x

Solution:

Given: f(x) = x e^xcosec x

$\frac{d}{dx}$ [f(x) g(x) h(x)] = f(x) g(x) h'(x) + f(x) g'(x) h(x) + f'(x) g(x) h(x)

f'(x)= cosec x[x (e^x)+ e^x (1)]+ x e^x [-Cosec x cot x]

= (x+1) e^xcosec x -x e^xcosec x cot x

f'(x) = e^xcosec x[x+1-x cot x]

7. Find the derivatives of the function if y = sin^5x

Solution:

Given: y = sin^5x

$y' = \frac{dy}{dx} = 5\sin^4 x \frac{d}{dx}(\sin x)$ = **5sin^4x cosx**

8. Find the derivatives of the function if y=sin (sin(sinx))

Solution:

Given y= sin (sin(sinx))

$y' = \frac{dy}{dx}$ = cos (sin(sin x)) cos(sinx) (cosx)

9. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then find $\frac{dy}{dx}$

Solution:

Given : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{y}{b^2}\frac{dy}{dx} = \frac{-x}{a^2}$ => $\mathbf{\frac{dy}{dx} = \frac{-b^2x}{a^2y}}$

10. Does the curve y = x^4- 2x^2 +2 have any horizontal tangents? If so where?

Solution:

Tangents are horizontal => $\frac{dy}{dx} = 0$

Given: y = x^4- 2x^2 +2

$\frac{dy}{dx} = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$

$\frac{dy}{dx} = 0$ => 4x(x-1)(x+1)=0

x=0, x=1, x=-1.

The curve will have horizontal tangents at (0,2), (1,1), (-1,1).

11. Find y' if y = x^x

Solution:

Given : y = x^x

Taking log on both sides

log y = logx^x

logy = xlogx

$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x}$ + logx (1) => $\frac{dy}{dx}$ = y [1 +logx] => **$\frac{dy}{dx}$ = x^x [1 +logx]**

12. Find y' if y = $(sinx)^x$

Solution:

Given: y = $(sinx)^x$

Taking log on both sides

log y = log$(sinx)^x$ => logy = x log (sinx)

$\frac{1}{y}\frac{dy}{dx} = x [\frac{1}{\sin x}\cos x]$ + logx (sinx) (1)

$\frac{dy}{dx}$ = y [xcotx + log (sinx)] => $\frac{dy}{dx}$ = $(\mathbf{sinx})^{\mathbf{x}}$**[xcotx + log (sinx)]**

13. Find y' if $y = (\log x)^{\sin x}$

Solution:

Given : $y = (\log x)^{\sin x}$

Taking log on both sides

$\log y = \log(\log x)^{\sin x}$

$\frac{1}{y}\frac{dy}{dx} = \sin x\,[\frac{1}{\log x}\frac{1}{x}] + \log(\log x)(\cos x)$

$\frac{dy}{dx} = y\,[\sin x\,[\frac{1}{\log x}\frac{1}{x}] + \log(\log x)(\cos x)]$

$\mathbf{\frac{dy}{dx} = (\log x)^{\sin x}\,[\sin x\,[\frac{1}{\log x}\frac{1}{x}] + \log(\log x)(\cos x)]}$

14. Find the domain of the function $f(x) = \frac{x+4}{x^2-9}$

Solution:

Given: $f(x) = \frac{x+4}{x^2-9}$ i.e., $y=\frac{x+4}{x^2-9}$

$x^2 - 9 = 0 \Rightarrow x = \pm 3$, division by zero is not allowed.

So **the domain is (-∞, -3) ∪ (-3,3) ∪ (3, ∞)**

15. Find the domain of the function $f(x) = \sqrt[3]{2x-1}$

Solution:

Given: $f(x) = \sqrt[3]{2x-1}$ i.e., $y = \sqrt[3]{2x-1} \Rightarrow y^3 = 2x - 1$

So **the domain is (-∞, ∞)**

16. Find the domain of the function $f(x) = \sqrt{3-x} - \sqrt{2+x}$

Solution:

Given: $f(x) = \sqrt{3-x} - \sqrt{2+x}$

$= 3-x \geq 0 \Rightarrow x \leq 3$

$2 + x \geq 0 \Rightarrow x \geq -2$

So, **the domain is -2 ≤ x ≤ 3 i.e., [-2, 3]**

PART B

1. Find the domain where function f is continuous. Also find the numbers at which the function f is discontinuous, where

$$f(x) = \begin{cases} 1 + x, & x \leq 0 \\ 2 - x, & 0 < x \leq 2 \\ (x-2)^2, & x > 2 \end{cases}$$

Ans: The domain of f is (-∞,0) ∪ (0,∞)

2. If $f(x) = \begin{cases} \frac{x^3-8}{x-2}, & x < 2 \\ ax^2 - bx + 3, & 2 \leq x < 3 \\ 2x - a + b, & x \geq 3 \end{cases} = \begin{cases} x^2 + 2x + 4, & x < 2 \\ ax^2 - bx + 3, & 2 \leq x < 3 \\ 2x - a + b, & x \geq 3 \end{cases}$ is continuous for all real x, find the values of a and b.

Ans: $\mathbf{a = \frac{-15}{2}}$ $\mathbf{b = \frac{-39}{2}}$

3. Find the equation of the tangent line to the parabola $y = x^2-8x+9$ at the point (3,-6)

Ans: y = -2x

4. Find dy/dx if $y = x^2e^x(x^2+1)^4$

Ans: dy/dx = $\mathbf{2xe^{2x}(x^2+1)^3[x^2+5x^2+x+1]}$

5. If e^x cosx = 1 = sin(xy), then find dy/dx

Ans: dy/dx = ycos (xy) + $\mathbf{e^y}$**sinx/**$\mathbf{e^y}$**cosx - x cos (xy)**

6. Find y if $(\sin x)^{\cos y} = (\sin y)^{\cos x}$

Ans: dy/dx = sinxlog (siny) + cosycotx/siny log (sinx) + cosx coty

7. If $y = (\cot x)^{\sin x} + (\tan x)^{\cos x}$, then find dy/dx

Ans: dy/dx = $\mathbf{(\cot x)^{\sin x}}$ **[-secx + cosx log (cotx)] +** $\mathbf{(\tan x)^{\cos x}}$ **[cosecx - sinx log tanx]**

8. Answer the following questions about the functions whose derivatives are given:

(a) What are the critical points of f?

(b) On what interval is f increasing or decreasing?

(c) At what points, if any, does f assume local maximum and minimum values?

(d) Find the interval of concavity and the inflection points.

(i) f(x) = sinx+cosx, $0 \le x \le 2\pi$

Ans: Inflection points are $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$ **since** $f\left(\frac{3\pi}{4}\right) = 0, f\left(\frac{7\pi}{4}\right) = 0$

(ii) $f(x) = x^4 - 2x^2 + 3$

Ans: Inflection points are $\left(\pm\frac{1}{\sqrt{3}}, \frac{22}{9}\right)$ **since** $f\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{22}{9}$

(iii) $f(x) = 2x^3 = 3x^2 - 36x$

Ans: Inflection points are $(-0.5, 18.5)$ **since** $f(-0.5) = 18.5$

10. Find the derivative of f(x) = cos $\left(\frac{b+acosx}{a+bcosx}\right)$

Ans: $f^{-1}(x) = \frac{-1}{\sqrt{1-\left(\frac{b+acosx}{a+bcosx}\right)^2}}\left[\frac{(a^2-b^2)sinx}{(a+bcosx)^2}\right]$

UNIT III – FUNCTIONS OF SEVERAL VARIABLES

PART –A

1. Evaluate: $\lim_{\substack{x\to1 \\ y\to2}} \frac{3x^2y}{x^2+y^2+5}$

Solution:

$$\lim_{\substack{x\to1 \\ y\to2}} \frac{3x^2y}{x^2+y^2+5} = \lim_{x\to1}\left[\lim_{y\to2} \frac{3x^2y}{x^2+y^2+5}\right] = \lim_{x\to1} \frac{3x^2(2)}{x^2+4+5}$$

$$= \lim_{x\to1} \frac{6x^2}{x^2+9} = \frac{6}{1+9} = \frac{6}{10} = \frac{3}{5}$$

2. If x= rcosθ, y = rsinθ, find (i) $\frac{\partial x}{\partial r}$ (ii) $\frac{\partial y}{\partial \theta}$ (iii) $\frac{\partial r}{\partial x}$ (iv) $\frac{\partial \theta}{\partial y}$

Solution:

Given : x= rcosθ y = rsinθ

(i) $\frac{\partial x}{\partial r} = \cos\theta$ (ii) $\frac{\partial y}{\partial \theta} = r\cos\theta$

(iii) $\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}$ $[r^2 = x^2 + y^2]$

$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$

(iv) $\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right]$ $[\theta = \tan^{-1}\frac{y}{x}]$

$\frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2}$

3. If u= $\frac{y}{z}+\frac{z}{x}$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

Solution:

Let u(x,y,z) = $\frac{y}{z}+\frac{z}{x}$

u(tx,ty,tz) = $\frac{ty}{tz}+\frac{tz}{tx} = t^0$ u(x,y,z)

⇨ n =0

=> u is a homogeneous function of x,y and z in degree 0.

By Euler's theorem, we get $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$ = nu=0.u = 0

4. Find $\frac{du}{dt}$ interms of t, if $u = x^3 + y^3$ where $x = at^2$, y =2at.

Solution:

$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$ ………………….. (1)

$u = x^3 + y^3$ => $\frac{\partial u}{\partial x} = 3x^2 = 3(at^2)^2 = 3a^2t^4$

=> $\frac{\partial u}{\partial y} = 3y^2 = 3(2at)^2 = 12a^2t^2$

$x = at^2 \Rightarrow \frac{dx}{dt} = 2at$, $y = 2at \Rightarrow \frac{dy}{dt} = 2a$

(1) $\Rightarrow \frac{du}{dt} = (3a^2t^4)(2at) + (12a^2t^2)(2a)$ $= \mathbf{6a^3t^5 + 24a^3t^2}$

5. If Z = f(y-z, z-x, x-y) show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$

Solution:

Let u = y-z, v = z-x, w = x-y

Z = f(u,v,w)

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1) \quad \Rightarrow \quad \frac{\partial Z}{\partial x} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

Similarly, $\frac{\partial Z}{\partial y} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$

$$\frac{\partial Z}{\partial z} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

Adding, $\mathbf{\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0}$

6. If x= rcosθ, y = rsinθ, find (i) $\frac{\partial(x,y)}{\partial(r,\theta)}$, (ii) $\frac{\partial(r,\theta)}{\partial(x,y)}$

Solution:

Given: x= rcosθ, y = rsinθ

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

(i) $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r[\cos^2\theta + \sin^2\theta] = \mathbf{r}$

(ii) We know that $\frac{\partial(x,y)}{\partial(r,\theta)}\frac{\partial(r,\theta)}{\partial(x,y)} = 1 \Rightarrow (r)\frac{\partial(r,\theta)}{\partial(x,y)} = 1$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = \mathbf{\frac{1}{r}}$$

7. If x = u (1 +v) and y = v(1+u) find $\frac{\partial(x,y)}{\partial(u,v)}$

Solution:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+u)(1+v) - uv = \mathbf{1+u+v}$$

8. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix} = -\frac{y^3}{2x^3} - \frac{y(x^2-y^2)}{2x^3} = \mathbf{-\frac{y}{2x}}$$

PART – B

1. If $g(x,y) = \varphi(u,v)$ where $u = x^2 - y^2$ and $v = 2xy$, then prove that $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 4(x^2+y^2)\left[\frac{\partial^2 \varphi}{\partial u^2} + \frac{\partial^2 \varphi}{\partial v^2}\right]$

2. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

3. If $u = f\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$

4. If u = 2xy, $v = x^2 - y^2$ and x = r cosθ, y = r sinθ. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution.

Ans: $\mathbf{\frac{\partial(u,v)}{\partial(r,\theta)} = -4r^3}$

5. If $x = u^2 - v^2$ and $y = 2uv$, then find the jacobian of x and y with respect to u and v.

Ans: $\mathbf{4(u^2 + v^2)}$

6. Find the jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3, if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Ans: 4

7. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

8. If x + y + z = u, y+z = uv, z = uvw, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$.

9. Find the extreme values of the function if f(x, y) = $x^3 + y^3 - 3x - 12y + 20$.

Ans: Maximum value = 38, Minimum value = 2

10. Examine $x^3 y^2 (12 - x - y)$ for extreme values.

Ans: Maximum Value = 6912

11. Find the maxima and minima of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Ans: Minimum Value = 8

12. A rectangular box open at the top, is to have a volume of 32 cc. find the dimensions of the box, that requires the least material for its construction.

Ans: Dimension of the box are 4, 4, 2

13. A rectangular box open at the top is to have a given capacity K. Find the dimensions of the box requiring least material for its construction.

Ans: Minimum value: 3 $\mathbf{(2K)^{2/3}}$

14. Find the greatest and the least distance of the point (3, 4, 12) from the unit sphere whose centre is at the origin.

Ans: Maximum distance = 14, Minimum distance =12

15. Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 square metre.

Ans: Dimension of the box are 12,12,6 and Maximum Volume = 864 cubic metres

16. A thin closed rectangular box is to have one edge equal to twice the other and constant volume $72m^3$. Find the least surface area of the box.

Ans: Minimum surface = 108

UNIT IV – INTEGRAL CALCULUS

PART – A

1. $\int_1^2 (x^3 - 2x)\, dx$

Solution:

Given: $\int_1^2 (x^3 - 2x)\, dx$, Here f(x)= $x^3 - 2x$

Antiderivative f(x) =F(x) =$\frac{x^4}{4} - \frac{2x^2}{2} = \frac{x^4}{4} - x^2$

A=$\int_{-1}^2 (x^3 - 2x)\, dx$ = F(b) - F(a) = F(2) - F(-1) by FTC2

$= \left[\frac{2^4}{4} - 2^2\right] - \left[\frac{(-1)^4}{4} - (-1)^2\right] = [4-4] - \left[\frac{1}{4} - 1\right] = \mathbf{\frac{3}{4}}$

2. Evaluate $\int \frac{x^3+2x+1}{x^4} dx$

Solution:

Given: $\int \frac{x^3+2x+1}{x^4} dx$

$= \int \left(\frac{1}{x} + \frac{2}{x^3} + \frac{1}{x^4}\right) dx = \int \left(\frac{1}{x} + 2x^{-3} + x^{-4}\right) dx$

$= \log x + \frac{2^{x^{-2}}}{(-2)} + \frac{x^{-3}}{(-3)} + C = \mathbf{\log x - \frac{1}{x^2} - \frac{1}{3x^3} + C}$

3. Evaluate $\int \left(\frac{6}{x^2} + \sqrt{x} + x^{3/2} + \frac{5}{x} + 1\right) dx$

Solution:

Given: $\int \left(\frac{6}{x^2} + \sqrt{x} + x^{3/2} + \frac{5}{x} + 1\right) dx$

$= \int \left(6x^{-2} + x^{1/2} + x^{3/2} + \frac{5}{x} + 1\right) dx$

$= 6\frac{x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + 5\log x + x + c$

$= -6\left(\frac{1}{x}\right) + \frac{2}{3}x^{3/2} + \frac{2}{5}x^{5/2} + 5\log x + x + c$

4. Evaluate $\int \frac{x^2+3x-5}{\sqrt{x}}dx$

Solution:

Given: $\int \frac{x^2+3x-5}{\sqrt{x}}dx$

$= \int \left[x^{2-\frac{1}{2}} + 3x^{1-\frac{1}{2}} - 5x^{-1/2}\right]dx = \int \left[x^{3/2} + 3x^{1/2} - 5x^{-1/2}\right]dx$

$= \frac{x^{5/2}}{(5/2)} + \frac{3x^{3/2}}{(3/2)} - \frac{5x^{1/2}}{(1/2)} + c = \frac{2}{5}x^{5/2} + 2x^{3/2} - 10x^{1/2} + c$

5. Evaluate $\int \frac{\sin^2 x}{1+\cos x}dx$

Solution:

$\int \frac{\sin^2 x}{1+\cos x}dx = \int \frac{1-\cos^2 x}{1+\cos x}dx$ $[\sin^2 x = 1 - \cos^2 x]$

$= \int \frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}dx$ $[a^2 - b^2 = (a-b)(a+b)]$

$= \int (1-\cos x)\,dx = x - \sin x + c$

6. Evaluate: $\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}dx$ (or) $\int_0^{\pi/2} \frac{1}{1+\tan x}dx$

Solution :

Let $I = \int_0^{\pi/2} \frac{\cos x\,dx}{\sin x+\cos x}$ $-(1)$

$= \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\pi/2-x\right)}dx$ $[\int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$I = \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}dx$ $-(2)$

(1) + (2)

$2I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}dx + \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}dx$

$= \int_0^{\frac{\pi}{2}} \left[\frac{\cos x}{\sin x+\cos x} + \frac{\sin x}{\cos x+\sin x}\right]dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\cos x+\sin x}dx$

$= \int_0^{\pi/2} 1\cdot dx = [x]_0^{\frac{\pi}{2}} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$

$\therefore I = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$

7. Evaluate: $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{199} x\,dx$

Solution:

Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^{199} x\,dx$

Let $f(x) = \sin^{199} x$

$f(-x) = \sin^{199}(-x) = [\sin(-x)]^{199}$

$= [-\sin x]^{199} = -\sin^{199} x = -f(x)$

$\therefore f(x)$ is an odd function

8. Evaluate $\int x(4+x^2)^{10}\,dx$

Solution:

Let $I = \int x(4+x^2)^{10}\,dx$

Put $u = 4 + x^2$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$

I = $\int u^{10} \frac{du}{2} = \frac{1}{2}\int u^{10}\, du = \frac{1}{2}\frac{u^{11}}{(11)} + c = \frac{1}{22}u^{11} + c$

$= \mathbf{\frac{1}{22}(4 + x^2)^{11} + c}$

9. Evaluate $\int (x+1)\sqrt{2x+x^2}\, dx$

Solution:

Let I = $\int (x+1)\sqrt{2x+x^2}\, dx$

Put $u = 2x + x^2$, $du = (2+2x)dx = 2(1+x)dx$, $\frac{du}{2} = (1+x)\, dx$

I = $\int \sqrt{u}\frac{du}{2} = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2}\frac{u^{3/2}}{(3/2)} + C = \frac{1}{3}u^{3/2} + c$

$= \mathbf{\frac{1}{3}\left(2x + x^2\right)^{3/2} + c}$

10. Evaluate $\int \frac{1}{x\log x} dx$

Solution:

Let I = $\int \frac{1}{x\log x} dx$

Put u = log x ; du=$\frac{1}{x}$dx

I = $\int \frac{1}{u} du = \log u + C = \mathbf{\log(\log x) + C}$

11. Evaluate $\int \frac{\sec^2(\log x)}{x} dx$

Solution:

Let I = $\int \frac{\sec^2(\log x)}{x} dx$

Put u = log x ; du = $\frac{1}{x}dx$

I = $\int \sec^2 u\, du = \tan u + c = \mathbf{\tan(\log x) + c}$

12. Evaluate $\int \cos^3\theta \sin\theta\, d\theta$

Solution:

Let I= $\int \cos^3\theta \sin\theta\, d\theta$

Put u = cosθ; du = - sinθ dθ;

I= $\int u^3(-du) = -\int u^3 du = -\frac{u^4}{4} + c = \mathbf{-\frac{\cos^4\theta}{4} + c}$

13. Evaluate $\int x^3 \cos(x^4+2)\, dx$

Solution:-

Let I = $\int x^3 \cos(x^4+2)\, dx$

Put $u=x^4+2$; $du=4x^3dx$; $du/4= x^3dx$

I= $\int \cos u \frac{du}{4} = \frac{1}{4}\sin u + C = \mathbf{\frac{1}{4}\sin(x^4+2) + C}$

14. Evaluate $\int x \sin x\, dx$

Solution:-

Let u = x dv = sinx dx [$\int u\, dv = uv - \int v\, du$]

du = dx v = $\int \sin x\, dx = -\cos x$

$\int x \sin x\, dx = (x)(-\cos x) - \int(-\cos x)dx$

$= -x\cos x + \int \cos x\, dx =$ **$-x\cos x + \sin x + C$**

15. Evaluate $\int(\log x)^2 dx$

Solution:-

Let u = $(\log x)^2$ dv=dx

du = 2logx (1/x)dx v=$\int dx = x$

$\int(\log x)^2 dx = (\log x)^2 x - \int x\, 2\log x(1/x)dx$ [$\int u\, dv = uv - \int v\, du$]

$= x(\log x)^2 - 2\int \log x\, dx$ ………..….(1)

Take $\int \log x \, dx$

Let $u = \log x$; $dv = dx$;

$du = 1/x \, dx$; $v = \int dx = x$

$\int \log x \, dx = (\log x)(x) - \int x \frac{1}{x} dx = x \log x - \int dx = x \log x - x$ $[\int u \, dv = uv - \int v \, du]$

(1)=> $\boldsymbol{\int (\log x)^2 dx = x(\log x)^2 - 2[x \log x - x] + C}$

16. For what values of p in the integral $\int_1^{\infty} \frac{1}{x^p} dx$ convergent?

Solution:-

If $p \neq 1$, $\lim_{t\to\infty} \int_1^t \frac{1}{x^p} dx = \lim_{t\to\infty} \int_1^t x^{-p} dx$

$= \lim_{t\to\infty} [x^{-p+1}/-p+1]_1^t$

$= \lim_{t\to\infty} [t^{-p+1}/-p+1 - 1/-p+1]$

$= \lim_{t\to\infty} 1/p-1[1-1/t^{p-1}]$

$$= \begin{cases} \frac{1}{p} - 1, & p > 1, \text{converges} \\ \infty, & p \leq 1, \text{diverges} \end{cases}$$

PART-B

1. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cot x}}{\sqrt{\cot x + \sqrt{\tan x}}} dx$ **Ans : I =** $\boldsymbol{\frac{\pi}{4}}$
2. Evaluate $\int_{-2}^{2} |x+1| dx$ **Ans: 5**
3. Evaluate $\int e^{\tan^{-1} x} \left[\frac{1+x+x^2}{1+x^2}\right] dx$ **Ans : I =** $\boldsymbol{x e^{\tan^{-1} x} + c}$
4. Using integration by parts, evaluate $\int \frac{(\ln x)^2}{x^2} dx$ **Ans:** $\boldsymbol{-\left(\frac{1}{x}\right)(\log x)^2 - 2\left[\frac{1}{x}\right] \log x - 2\left[\frac{1}{x}\right] + c}$
5. Evaluate $\int (\log x)^2 dx$ **Ans:** $\boldsymbol{x(\log x)^2 - 2[x \log x - x] + c}$
6. Evaluate $\int \frac{1}{\sqrt{a^2 - x^2}} dx$ by using trigonometric substitution **Ans: I =** $\boldsymbol{\sin^{-1}\left(\frac{x}{a}\right) + c}$
7. Integrate $\frac{1}{x^2+3x+2}$ **Ans:** $\boldsymbol{\log\left(\frac{x+1}{x+2}\right) + c}$
8. Integrate $\frac{x^2+1}{(x^2-1)(2x+1)}$ **Ans:** $\boldsymbol{\frac{1}{3}\log(x-1) + \log(x+1) - \frac{5}{6}\log(2x+1) + c}$
9. Integrate $\int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2} dx$ **Ans:** $\boldsymbol{\log\left(\frac{1+\tan x}{2+\tan x}\right) + c}$
10. Evaluate $\int_3^{\infty} \frac{1}{(x-2)^{3/2}} dx$ **Ans:** $\boldsymbol{\int_3^{\infty} \frac{1}{(x-2)^{3/2}}}$ **dx is convergent**
11. Sketch the region bounded by the curves and visually estimate the location of the centroid. Then find the exact co-ordinates of the centroid $y = \sqrt{x}$, $y=0$, $x = 4$ **Ans: The Centroid of the region is (2.4, 0.75)**
12. Sketch the region bounded by the curves and visually estimate the location of the centroid. Then find the exact co-ordinates of the centroid $y = \sin x$, $y = 0$, $0 \leq x \leq \pi$ **Ans: The Centroid of the region is** $\boldsymbol{\left(\frac{\pi}{2}, \frac{\pi}{8}\right)}$
13. Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$ **Ans: The Centroid is** $\boldsymbol{\left(\frac{1}{2}, \frac{2}{5}\right)}$

UNIT V – MULTIPLE INTRGRALS

PART –A

1. Evaluate $\int_0^1 \int_1^2 x(x+y) dy \, dx$

Solution:-

$\int_0^1 \int_1^2 x(x+y) dy \, dx = \int_0^1 \int_1^2 [x^2+xy] dy \, dx$

$= \int_0^1 [x^2 y + \frac{xy^2}{2}]_{y=1}^{y=2} dx$

$= \int_0^1 [(2x^2+2x) - (x^2 + \frac{x}{2})] dx$

$= \int_0^1 [2x^2 + 2x - x^2 - \frac{x}{2}] dx = \int_0^1 [x^2 + (3/2) x] dx$

$= [\frac{x^3}{3} + (3/2)(\frac{x^2}{2})]_0^1 = (\frac{1}{3} + \frac{3}{4}) - (0+0)$

$= \boldsymbol{\frac{13}{12}}$

2. Evaluate: $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

Solution:-

Let I = $\int_0^{\pi/2}\int_0^{\sin\theta} r\,d\theta\,dr$

$= \int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$ [correct form]

$=\int_0^{\Pi/2}[r^2/2]_0^{\sin\theta}\,d\theta$

$=\int_0^{\pi/2}[\frac{(\sin\vartheta)^2}{2} - 0]\,d\theta$

$= \frac{1}{2}\int_0^{\pi/2} sin^2\vartheta\,d\theta$

$= \frac{1}{2}.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{8}$

3. Evaluate $\int_0^{\Pi}\int_0^{\sin\theta} r\,dr\,d\theta$

Solution:-

Let I = $\int_0^{\Pi}\int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\Pi}[r^2/2]_0^{\sin\theta}\,d\theta$

$= \int_0^{\Pi} \sin^2\theta/2\,d\theta$

$= 1/2 \int_0^{\Pi}[1-\cos2\theta/2]d\theta$

$=1/4\,[\theta-\sin2\theta/2)_0^{\Pi}$

$=1/4\,[(\Pi-0)-(0-0)]$

$= \boldsymbol{\frac{\pi}{4}}$

4. Evaluate $\int_0^1\int_0^{1-x} y\,dy\,dx$

Solution:-

Let I = $\int_0^1\int_0^{1-x} y\,dy\,dx = \int_0^1[y^2/2]_{y=0}^{y=1-x}dx$

$= \int_0^1[(1-x)^2/2 - 0]dx = \int_0^1(1-x)^2/2\,dx$

$= ½\int_0^1(1-x)^2dx = ½\,[(1-x)^3/3(-1)]_0^1$

$= -1/6[(1-x)^3]_0^1 = -1/6[0-1] =$ **1/6**

5. Find the area of a circle of radius 'a' by double integration in polar co-ordinates.

Solution:

The equation of circle with pole on the circle and diameter through the point as initial line is r=2a cosθ

Area = 2 **X** upper area

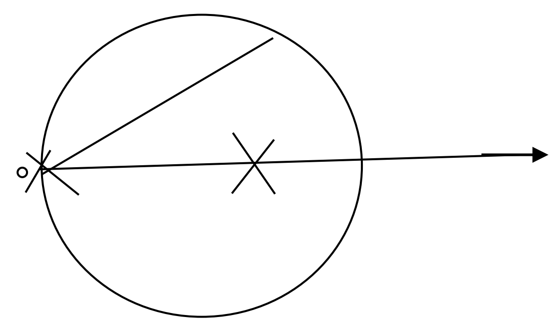
$= 2\int_0^{\Pi/2}\int_0^{2a\cos\theta} r\,dr\,d\theta$

$= \int_0^{\Pi/2}(r^2)_0^{2a\cos\theta}\,d\theta$

$= 4a^2\int_0^{\Pi/2}\cos^2\theta\,d\theta$

$=4a^2(1/2)(\Pi/2)$

$= \Pi a^2$ square units.



PART –B

1. Evaluate $\int_0^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$ **Ans: 2 + 8(log8 -2)**

2. Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} r\,d\,\theta\,dr$ **Ans: I =** $\boldsymbol{\frac{\pi}{8}}$

3. Change the order of integration in $\int_0^a\int_x^a(x^2 + y^2)\,dydx$ and hence evaluate it **Ans:** $\boldsymbol{\frac{a^4}{3}}$

4. Change the order of integration in $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ and hence evaluate it **Ans:** $\boldsymbol{\frac{3}{8}}$

5. Change the order of integration in $\int_0^1\int_x^{2-x}\frac{x}{y}dy\,dx$ and hence evaluate it **Ans: log 4-1**

6. Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$ **Ans:** $\boldsymbol{A=\frac{1}{8}(2a^2 - 1)}$

7. Show that area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $\frac{16}{3}a^2$

8. Find using double integral, the area of cardioid r = a(1 + cosθ) **Ans:** $\boldsymbol{\frac{3}{2}a^2\pi}$ **square units**

9. Calculate $\iint r^3\,dr\,d\theta$ over the area included between the circles r = 2sinθ and r = 4sinθ **Ans:** $\boldsymbol{\frac{45}{2}\pi}$ **square units**

10. Find the area that lies inside the cardioid r = a (1 + cosθ) and outside the circle r = a, by double integration

Ans: $\frac{a^2}{4}(\pi + 8)$ **square units**

11. Evaluate: I = $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx$ **Ans:** $I = \frac{8}{3}\log 2 - \frac{19}{9}$

12. Evaluate: $\int_0^{\log a}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$ **Ans:** $I = \frac{1}{8}[a^4 - 6a^2 + 8a - 3]$

13. Evaluate: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$ **Ans:** $\frac{\pi^2}{8}$

14. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and planes y + 4 and z = 0 **Ans: 16π cubic units**

15. Find the volume of sphere $x^2 + y^2 + z^2 = a^2$ without transformation **Ans:** $\frac{4}{3}\pi a^3$ **cubic units**

16. Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Ans: $\frac{abc}{6}$ **cubic units**

17. Find the volume of that portion of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which lies in the first octant using triple integration

Ans: $\frac{\pi\, abc}{6}$ **cubic units**

18. Evaluate: $\iiint \frac{dz\, dy\, dx}{(x+y+z+1)^3}$ over the region of integration bounded by the planes x = 0, y = 0, x + y + z = 1.

Ans: $\frac{1}{2}\log 2 - \frac{5}{16}$

19. Evaluate: $\iiint \frac{dz\, dy\, dx}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ over the first octant of sphere $x^2 + y^2 + z^2 = a^2$ **Ans:** $\frac{\pi^2 a^2}{8}$

20. By changing to polar co-ordinates, find the value of integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2 + y^2)\, dy\, dx$

Ans: $\frac{3\pi}{4}a^4$

21. By changing into polar co-ordinates show that $\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \frac{\pi}{4}$. Hence evaluate $\int_0^{\infty} e^{-t^2}\, dt$ **Ans:** $\frac{\sqrt{\pi}}{2}$

22. Evaluate by changing into polars, the integral $\int_0^{a}\int_y^{a}\frac{x^2}{\sqrt{x^2+y^2}}\, dxdy$ **Ans:** $\frac{a^3}{3}[\log(\sqrt{2}+1)]$

23. Evaluate $\int\int\int \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dxdy$ over the region bounded by the sphere $x^2 + y^2 + z^2 = 1$. **Ans:** π^2

24. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dzdydx}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar co-ordinates. **Ans:** $\frac{\pi^2}{8}$

25. Evaluate the integration $\int\int\int xyzdxdydz$ taken throughout the volume for which x, y, z ≥0 and $x^2 + y^2 + z^2 \le 9$

Ans: $\frac{243}{16}$