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EC8451 - ELECTROMAGNETIC FIELDS

UNIT - I INTRODUCTION

Electromagnetic model, units and constants, review of vector algebra, Rectangular, cylindrical and spherical coordinate systems, Line, Surface and volume integrals, Gradient of a scalar field, Divergence of a vector field, Divergence theorem, curl of a vector field, Stoke's theorem, Null identities, Helmholtz's theorem.

Electromagnetic model :

Time varying electric and magnetic field is called as Electro magnetic field.

Fundamental electromagnetic field quantities

1. Electric field intensity (EFI) : Symbol → E
   Unit → V/m
2. Electric flux density : Symbol → D
   Unit → C/m^2
3. Magnetic field intensity (MFI) : Symbol → H
   Unit → A/m
4. Magnetic flux density : Symbol → B
   Unit → T (Tesla)

* Units and constants :

Units : Example → m, kg, s, A

constants : Example → $c = 3 \times 10^8$ m/s

$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m

$\mu_0 = 4\pi \times 10^{-7}$ H/m

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Review of vector Algebra :-

* **Scalar :**

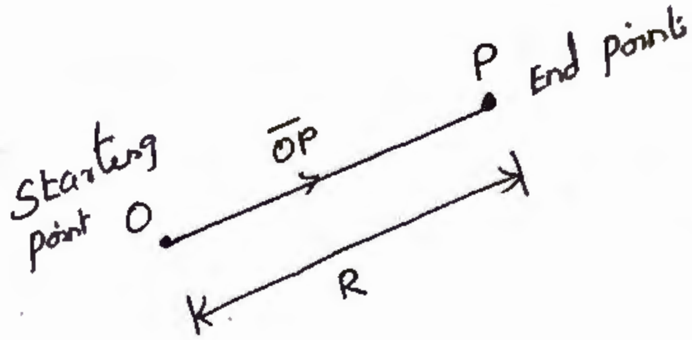
Scalar is a physical quantity which has only magnitude. Ex: Mass, charge, work and temperature

* **Vector :-**

vector is a physical quantity which has both magnitude and direction.

Ex: velocity, Force, electric field and magnetic field.

* **Representation of vector ($\bar{OP}$) :-**



$$\text{vector} = \text{Magnitude} \times \text{direction}$$

$$\text{Length of } OP = |\vec{OP}| \text{ (or) } |\vec{R}| = R$$

$$\therefore R = \text{End point} - \text{Starting point}$$

$$R = P - O$$

Unit vector ($\hat{OP}$ or $\hat{a}_{op}$ or $\hat{a}_R$) :-

$$\hat{a}_{op} = \frac{\vec{OP}}{|\vec{OP}|} \quad \text{(or)} \quad \hat{a}_R = \frac{\vec{R}}{|\vec{R}|}$$

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* Example :-

$$\bar{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$$

$$\text{Magnitude of } \vec{A} = |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\text{unit vector of } \vec{A} = \hat{a} = \frac{\bar{A}}{|\bar{A}|}$$

* Sum and difference of two vectors :-

$$\vec{A} + \vec{B} = (A_x + B_x)\vec{a_x} + (A_y + B_y)\vec{a_y} + (A_z + B_z)\vec{a_z}$$

$$\vec{A} - \vec{B} = (A_x - B_x)\vec{a_x} + (A_y - B_y)\vec{a_y} + (A_z - B_z)\vec{a_z}$$

* Dot product :-

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

If $\theta = 0^\circ$ (parallel vector), $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|$

If $\theta = 90^\circ$ (perpendicular), $\vec{A} \cdot \vec{B} = 0$

* Dot product of two vector is a Scalar.

* Cross product :-

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\theta \, \vec{a_n}$$

where

$\vec{a_n}$ = unit vector $\perp$r to $\bar{A}$ and $\bar{B}$

$$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

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If $\theta = 0°$ (Parallel), $\vec{A} \times \vec{B} = 0$

If $\theta = 90°$ (Perpendicular), $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\vec{a}_n$

(or)

unit vector normal, $\vec{a}_n = \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$

where $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

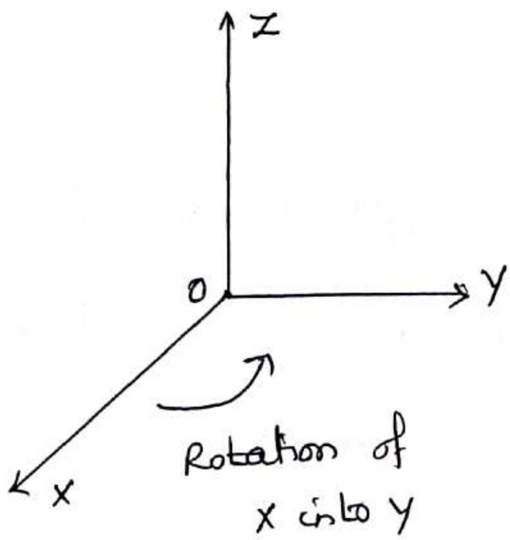
Coordinate Systems

Coordinate system is defined as a system which is used to represent a point space.

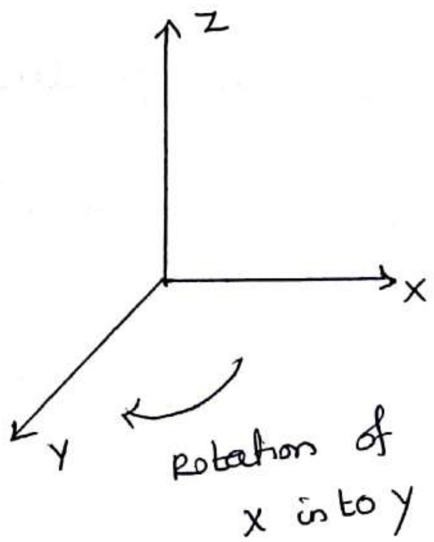
Types of commonly used coordinate systems are:

1. Cartesian (or) Rectangular coordinate system
2. Cylindrical (or) circular cylindrical coordinate system
3. Spherical coordinate system

* **Cartesian (or) Rectangular coordinate system :-**



a) Right handed system



b) Left handed system

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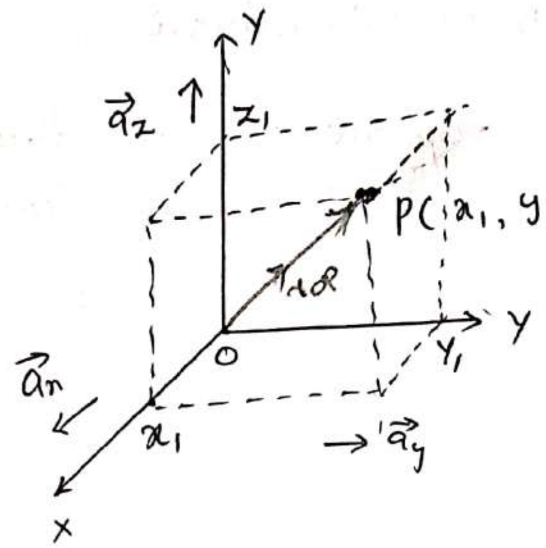
* Defining a point (P) in cartesian coordinate system:-



x – Width in metre (m)

y – Length in metre

z – Height in metre

Point P having coordinates x_1, y_1 and z_1

It is represented by $P(x_1, y_1, z_1)$

Unit vectors (or) base vectors : $\hat{a}_x$, $\hat{a}_y$ and $\hat{a}_z$

position vector (or) radius vector : line connecting from origin to point P.

$$\vec{r}_{op} = x_1 \vec{a}_x + y_1 \vec{a}_y + z_1 \vec{a}_z$$

Length (or) magnitude of OP :

$$|\vec{OP}| = r_{op} = \sqrt{(x_1)^2 + (y_1)^2 + (z_1)^2}$$

* The ranges of cartesian variables are :

x varies from $-\infty < x < \infty$

y varies from $-\infty < x < \infty$

z varies from $-\infty < x < \infty$

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* Infinitesimal elements (or) differential elements in cartesian coordinate system:-

* infinitesimal displacement (or) differential displacement (or) differential vector length:-

$$\boxed{\vec{dl} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}}$$

where

dx = Small differential length in x direction
dy = Small differential length in y direction
dz = Small differential length in z direction

* Infinitesimal volume (or) differential volume

$$\boxed{dv = dx\,dy\,dz}$$

* Infinitesimal Surface (or) differential normal Surface:

$$\boxed{\begin{aligned} ds_x &= dy\,dz\,\vec{a_x} \\ ds_y &= dx\,dz\,\vec{a_y} \\ ds_z &= dx\,dy\,\vec{a_z} \end{aligned}}$$

where

ds_x = differential Surface area normal to $\vec{a_x}$
ds_y = differential Surface area normal to $\vec{a_y}$
ds_z = differential Surface area normal to $\vec{a_z}$

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* Vector components in Cartesian coordinate system :-

Dot produce of unit vectors :

$$\vec{a_x} \times \vec{a_y} = \vec{a_z} = -\vec{a_y} \times \vec{a_x}$$

$$\vec{a_y} \times \vec{a_z} = \vec{a_x} = -\vec{a_y} \times \vec{a_z}$$

$$\vec{a_z} \times \vec{a_x} = \vec{a_y} = -\vec{a_z} \times \vec{a_x}$$

Cross product of unit vectors :

$$\vec{a_x} \cdot \vec{a_x} = \vec{a_y} \cdot \vec{a_y} = \vec{a_z} \cdot \vec{a_z} = 1$$

$$\vec{a_x} \cdot \vec{a_y} = \vec{a_y} \cdot \vec{a_z} = \vec{a_z} \cdot \vec{a_x} = 0$$

Ex:

$$\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}$$

$$\vec{B} = B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z}$$

$$\vec{A} \cdot \vec{B} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) \cdot (B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z})$$

$$= A_x B_x + A_y B_y + A_z B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \vec{a_x}(A_y B_z - B_y A_z) + \vec{a_y}(A_x B_z - B_x a_z)$$

$$+ \vec{a_z}(A_x B_y - A_y B_x)$$

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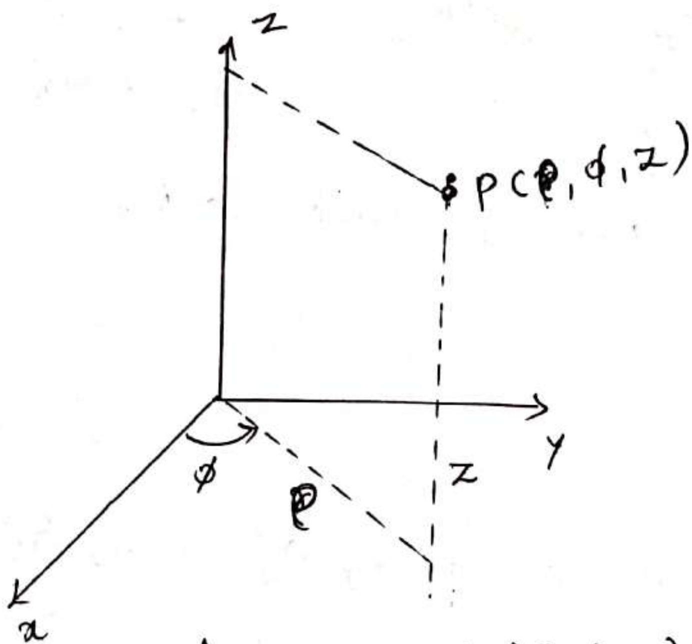
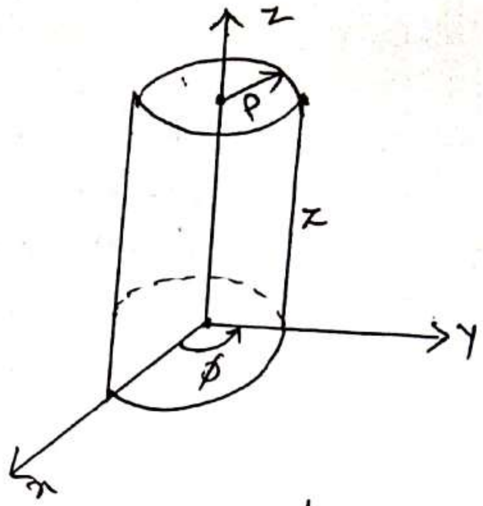
* Circular cylindrical coordinate system :-

The circular cylindrical coordinate system is the three dimensional version of polar coordinate system.



Defining point P(ρ, φ, z)



cylindrical surface along z-axis

Point P having coordinates ρ, φ and z. It is represented by P(ρ, φ, z)

unit vectors : $\vec{a}_\rho$, $\vec{a}_\phi$ and $\vec{a}_z$

position vectors : $\vec{A} = A_\rho \vec{a}_\rho + A_\phi \vec{a}_\phi + A_z \vec{a}_z$

* The ranges of cylindrical variables are :

ρ varies from $0 \le \rho \le \infty$

φ varies from $0 \le \phi \le 2\pi$

z varies from $-\infty \le z < \infty$

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* Infinitesimal elements (or) differential elements in cylindrical coordinate system:

* Infinitesimal displacement (or) differential displacement (or) differential vector length:

$$\boxed{\vec{dl} = d\rho\, \vec{a_\rho} + \rho\, d\phi\, \vec{a_\phi} + dz\, \vec{a_z}}$$

where $d\rho$ = differential length in r - direction
$\rho d\phi$ = differential length in ϕ - direction
dz = differential length in z - direction

* Infinitesimal volume (or) differential volume:

$$\boxed{dv = \rho\, d\rho\, d\phi\, dz}$$

* Infinitesimal surface (or) differential normal surface:

$$\vec{ds_\rho} = \rho\, d\phi\, dz\, \vec{a_\rho}$$

$$\vec{ds_\phi} = d\rho\, dz\, \vec{a_\phi}$$

$$\vec{ds_z} = \rho\, d\rho\, d\phi\, \vec{a_z}$$

where ds_ρ = differential surface area normal to $\vec{a_\rho}$
ds_ϕ = differential surface area normal to $\vec{a_\phi}$
ds_z = differential surface area normal to $\vec{a_z}$

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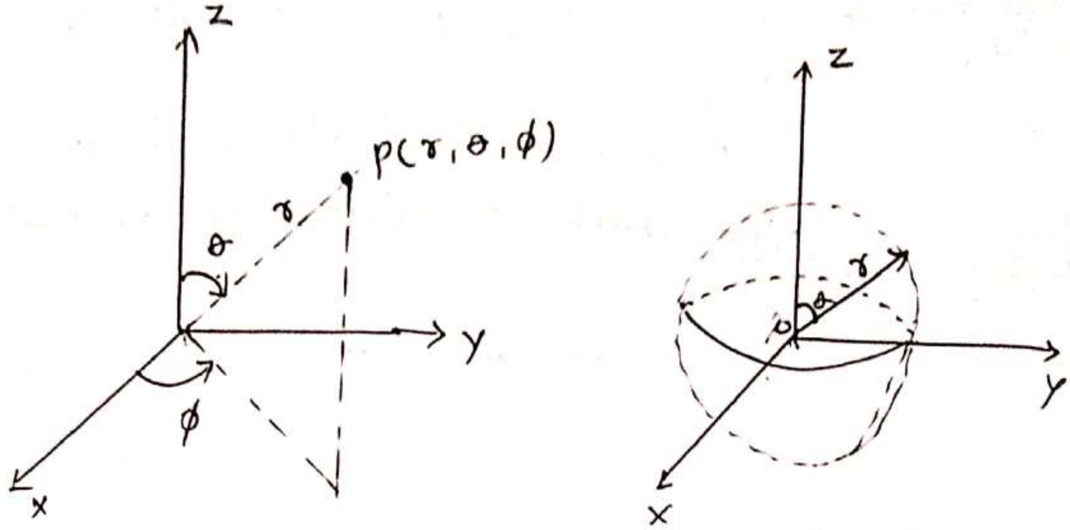
* Spherical coordinate System :-



Position of point P in spherical coordinate system

Spherical Surface

Point P having coordinates r, θ and ϕ

It is represented by P(r, θ, ϕ)

Unit vectors : $\vec{a}_r$, $\vec{a}_\theta$ and $\vec{a}_\phi$

Position vector : $\bar{P} = P_r \vec{a}_r + P_\theta \vec{a}_\theta + P_\phi \vec{a}_\phi$

* The ranges of Spherical variables are

r varies from $0 \le r \le \infty$

θ varies from $0 \le \theta \le \pi$

ϕ varies from $0 \le \phi \le 2\pi$

* Infinitesimal elements (or) differential elements in spherical coordinate System :-

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* infinitesimal displacement (or) differential displacement (or) differential vector length ⑪

$$\boxed{\vec{dl} = dr\,\vec{a_r} + r\,d\theta\,\vec{a_\theta} + r\sin\theta\,d\phi\,\vec{a_\phi}}$$

* Infinitesimal volume (or) differential volume

$$\boxed{dv = r^2 \sin\theta\, dr\, d\theta\, d\phi}$$

* Infinitesimal Surface (or) differential normal Surface :-

$$ds_r = r^2\, d\theta\, d\phi\, \sin\theta\, \vec{a_r}$$

$$ds_\theta = r\sin\theta\, dr\, d\phi\, \vec{a_\theta}$$

$$ds_\phi = r\, dr\, d\theta\, \vec{a_\phi}$$

* Relationship between different coordinate System :-

i) cylindrical to cartesian

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

ii) cartesian to cylindrical

$$\rho = \sqrt{x^2+y^2}$$

$$\phi = \tan^{-1} y/x$$

$$z = z$$

iii) Spherical to cartesian

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

iv) Cartesian to Spherical

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\phi = \tan^{-1} y/x$$

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* Transformation of vectors :-

i) Cartesian to circular cylindrical :-

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$A_\rho = A_x \cos\phi + A_y \sin\phi$; $A_\phi = -A_x \sin\phi + A_y \cos\phi$

$A_z = A_z$

ii) circular cylindrical to cartesian :-

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$A_x = \cos\phi\, A_\rho - A_\phi \sin\phi$; $A_y = A_\rho \sin\phi + A_\phi \cos\phi$; $A_z = A_z$

iii) cartesian to spherical :-

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\theta & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

iv) Spherical to cartesian :-

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

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* vector differential operator (∇) :-

vector differential operator is denoted by ∇.

It is also known as del operator.

For cartesian coordinate system,

$$\nabla = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}$$

For cylindrical coordinate system,

$$\nabla = \frac{\partial}{\partial \rho}\vec{a_\rho} + \frac{1}{r}\frac{\partial}{\partial \phi}\vec{a_\phi} + \frac{\partial}{\partial z}\vec{a_z}$$

For spherical coordinate system,

$$\nabla = \frac{\partial}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\vec{a_\phi}$$

---

* Gradient of a Scalar field (∇V) :-

The operation of the vector differential operation (∇) on a scalar function (V) is called Gradient of a scalar.

It gives the maximum space rate of change of the Scalar.

Gradient of Scalar is a vector.

For cartesian coordinates,

$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

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For cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}$$

For spherical coordinates,

$$\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}$$

---

* <u>Divergence of a vector field</u> :- $(\nabla \cdot \bar{D})$

Divergence is defined as the net outward flow of the flux per unit volume over a closed incremental surface

It is denoted as Div $\vec{D}$ (or) $\nabla \cdot \vec{D}$ (or) Divergence of $\vec{D}$.

$$\text{Div}\,\vec{D} = \nabla\cdot\vec{D} = \lim_{\Delta v \to 0} \frac{\oint \vec{D}\cdot d\vec{s}}{\Delta v}$$

Divergence of a vector function is a <u>Scalar</u>.

For cartesian coordinate system,

$$\nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

For cylindrical coordinate system,

$$\nabla\cdot\vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial D_z}{\partial z}$$

For Spherical coordinate system,

$$\nabla\cdot\vec{D} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(D_\phi)$$

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* curl of a vector field $(\nabla \times \vec{H})$:-

The curl of a vector field $\vec{H}$ is defined as a vector whose magnitude is the maximum net circulation of $\vec{H}$ per unit area as area tends to zero.

It is denoted by curl $\vec{H}$ (or) $\nabla \times \vec{H}$
curl of a vector function is a vector.

$$\text{curl } \vec{H} = \nabla \times \vec{H} = \lim_{\Delta s \to 0} \frac{\oint_s \vec{n} \times \vec{H}\, d\vec{s}}{\Delta s}$$

For cartesian coordinate system,

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix}$$

For cylindrical coordinate system,

$$\nabla \times \vec{H} = \frac{1}{\rho} \begin{vmatrix} \vec{a_\rho} & \rho\vec{a_\phi} & \vec{a_z} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

For spherical coordinate system,

$$\nabla \times \vec{H} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{a_r} & r\vec{a_\theta} & r\sin\theta\, \vec{a_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ H_r & rH_\theta & r\sin\theta\, H_\phi \end{vmatrix}$$

* Solenoidal field : $\nabla \cdot \vec{D} = 0$

Irrotational field : $\nabla \times \vec{H} = 0$

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Divergence Theorem (or) Gauss Divergence theorem (or) Gauss astrogradsky Theorem :- (16)

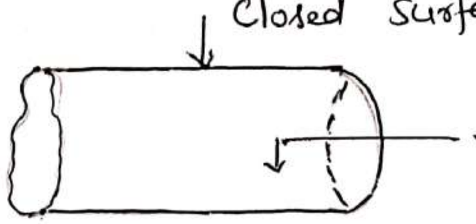
Statement :-

" The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that closed surface".

$$\boxed{\oint_S \vec{D} \cdot \vec{ds} = \oint_V (\nabla \cdot \vec{D})\, dv} \rightarrow (1)$$

It converts surface integral in to volume integral.

Proof :-



Let us consider equation for charge enclosed in volume Δv.

$$\Delta Q = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) . \Delta x \Delta y \Delta z \rightarrow (2)$$

The total outward flux is equal to the charge enclosed

$$\Delta \psi = \Delta Q = \rho_v \, dv = \rho_v \, \Delta x \Delta y \Delta z \rightarrow (3)$$

By comparing eq's (2) and (3),

$$\rho_v = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \rightarrow (4)$$

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Now the vector differential operator ∇ is (17)

$$\nabla = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z} \rightarrow (5)$$

The divergence of $\bar{D}$ is given by

$$\nabla\cdot\vec{D} = \left(\frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}\right)\cdot(D_x\vec{a_x} + D_y\vec{a_y} + D_z\vec{a_z})$$

$$= \frac{\partial D_x}{\partial x}(\vec{a_x}\cdot\vec{a_x}) + \frac{\partial D_x}{\partial y}(\vec{a_y}\cdot\vec{a_x}) + \frac{\partial D_x}{\partial z}(\vec{a_z}\cdot\vec{a_x}) + \frac{\partial D_y}{\partial x}(\vec{a_x}\cdot\vec{a_y}) + \frac{\partial D_y}{\partial y}(\vec{a_y}\cdot\vec{a_y}) + \frac{\partial D_z}{\partial x}(\vec{a_x}\cdot\vec{a_z}) + \frac{\partial D_z}{\partial y}(\vec{a_y}\cdot\vec{a_z}) + \frac{\partial D_z}{\partial z}(\vec{a_z}\cdot\vec{a_z}) \rightarrow (6)$$

By using dot product rule,

$$\vec{a_x}\cdot\vec{a_x} = \vec{a_y}\cdot\vec{a_y} = \vec{a_z}\cdot\vec{a_z} = 1$$

$$\vec{a_x}\cdot\vec{a_y} = \vec{a_y}\cdot\vec{a_z} = \vec{a_z}\cdot\vec{a_x} = 0$$

$$\therefore \nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \rightarrow (7)$$

comparing eq's (4) and (7)

$$\nabla\cdot\vec{D} = \rho_v \rightarrow (8) \quad \text{(Point form of Gauss's law)}$$

Now according to integral form Gauss's law

$$\oint_s \vec{D}\cdot\vec{ds} = \oint_v \rho_v\, dv \rightarrow (9)$$

Substitute eq (8) in to eq (9)

$$\boxed{\oint_s \vec{D}\cdot\vec{ds} = \oint_v (\nabla\cdot\vec{D})\, dv}$$

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* **Stoke's theorem :-**

Statement :-

"The integration of any vector around a closed path is always equal to the integration of curl of that vector throughout the surface enclosed by that path".

$$\boxed{\oint_l \vec{H} \cdot \vec{dl} = \oint_S (\nabla \times \vec{H}) \cdot \vec{ds}} \rightarrow (1)$$

Proof :-

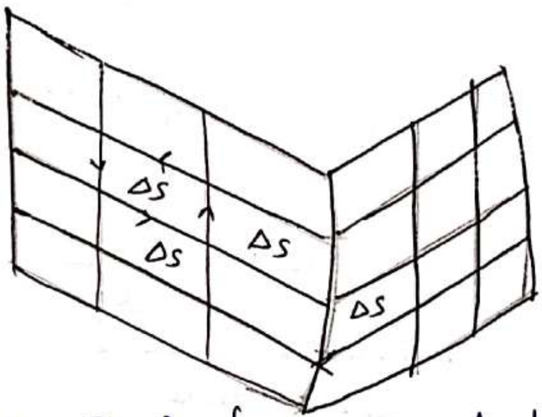

Fig (a): Surface 'S' broken into incremental Surface ΔS.

Consider a surface S which is broken into the incremental surface ΔS as shown in figure (a).

According to Ampere's circuital law, the current enclosed by the closed surface is given by

$$\oint_l \vec{H} \cdot \vec{dl} = I \rightarrow (2)$$

Now current I and current density J are related as

$$I = \oint_S \vec{J} \cdot \vec{ds} \rightarrow (3)$$

Substitute eq (3) into eq (2),

$$\oint_l \vec{H} \cdot \vec{dl} = \oint_S \vec{J} \cdot \vec{ds} \rightarrow (4)$$

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According to point form of Ampere's law

$$\nabla \times \vec{H} = \vec{J} \rightarrow (5)$$

Substitute eq (5) in to (4),

$$\boxed{\oint_{l} \vec{H} \cdot d\vec{l} = \oint_{s} (\nabla \times \vec{H}) \cdot d\vec{s} \rightarrow (6)}$$

It converts line integral in to surface integral.

* **Null Identities** :-

i) $\nabla \times (\nabla V) = 0$ ⇒ Curl of Gradient of any Scalar field V is zero

2) $\nabla \cdot (\nabla \times \vec{A}) = 0$ ⇒ Divergence of curl of any vector field is zero

* **HELMHOLTZ'S THEOREM** :-

The vectors fields are classified in accordance with their nature of field either solenoidal and/or irrotational.

i) Solenoidal : $\nabla \cdot \vec{F} = 0$ and Irrotational : $\nabla \times \vec{F} = 0$

Ex: A static electric field in a charge free region

ii) Solenoidal but not irrotational :

$\nabla \cdot F = 0$ and $\nabla \times \vec{F} \neq 0$

Ex: A steady magnetic field in current-carrying conductor.

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iii) Irrotational but not Solenoidal :

$$\nabla \times \vec{F} = 0 \quad \text{and} \quad \nabla \cdot \vec{F} \neq 0$$

Example : A static electric field in a charged region

iv) Neither solenoidal nor Irrotational :

$$\nabla \cdot \vec{F} \neq 0 \quad \text{and} \quad \nabla \times \vec{F} \neq 0$$

Ex : An electric field in a charged medium with a time varying magnetic field.

Theorem :-

Helmholtz's theorem states that a general vector function F can be written as the sum of the gradient of a scalar function and the curl of a vector function.

$$\boxed{\vec{F} = -\nabla V + \nabla \times \vec{A}}$$

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UNIT - I

2 Marks Q & A

1. ① State the fundamental theorem of divergence.

Gauss Divergence Theorem :-

" The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by that surface"

$$\therefore \oint_S \vec{D} \cdot \vec{ds} = \oint_V \nabla \cdot \vec{D} \, dv$$

② Write Stoke's theorem in integral form.

" The integration of any vector around a closed path is always equal to the integration of curl of that vector through the surface enclosed by that path ".

$$\therefore \oint_l \vec{H} \cdot \vec{dl} = \oint_S (\nabla \times \vec{H}) \cdot \vec{ds}$$

③ Write the infinitesimal displacement, spherical and cylindrical coordinates.

Cartesian coordinate system :

$$\vec{dl} = dx \, \vec{a_x} + dy \, \vec{a_y} + dz \, \vec{a_z}$$

$$ds_x = dy \, dz \, \vec{a_x}$$

$$ds_y = dx \, dz \, \vec{a_y}$$

$$ds_z = dx \, dy \, \vec{a_z}$$

$$dv = dx \, dy \, dz$$

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cylindrical coordinate system :-

$$d\vec{l} = d\rho\, \vec{a_\rho} + r\, d\phi\, \vec{a_\phi} + dz\, \vec{a_z}$$

$$dv = \rho\, d\rho\, d\phi\, dz$$

$$ds_\rho = \rho\, d\phi\, dz\, \vec{a_\rho}$$

$$ds_\phi = d\rho\, dz\, \vec{a_\phi}$$

$$ds_z = \rho\, d\rho\, d\phi\, \vec{a_z}$$

Spherical coordinate system :-

$$d\vec{l} = dr\, \vec{a_r} + r\, d\theta\, \vec{a_\theta} + r \sin\theta\, d\phi\, \vec{a_\phi}$$

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$ds_r = r^2\, d\theta\, d\phi\, \sin\theta\, \vec{a_r}$$

$$ds_\theta = r \sin\theta\, dr\, d\phi\, \vec{a_\theta}$$

$$ds_\phi = r\, dr\, d\theta\, \vec{a_\phi}$$

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(4) Define Gradient of Scalar field.

It gives the maximum space rate of change of the Scalar.

Gradient of a Scalar is a vector.

In cartesian coordinates

$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

In cylindrical coordinates,

$$\nabla V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}$$

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In Spherical coordinates

$$\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}$$

(5) Define divergence of vector field.

Divergence is defined as the net outward flow of the flux per unit volume over a closed incremental surface.

$$\nabla \cdot \vec{D} = \lim_{\Delta v \to 0} \frac{\oint \vec{D} \cdot d\vec{s}}{\Delta v}$$

$$\nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \rightarrow \text{Cartesian coordinates}$$

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(D_\phi) + \frac{\partial D_z}{\partial z}$$

↳ Cylindrical coordinates

$$\nabla \cdot \vec{D} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(D_\phi).$$

↳ Spherical coordinates

(6) Define curl of a vector field.

It is defined as the maximum net circulation of $\vec{H}$ per unit area.

$$\nabla \times \vec{H} = \lim_{\Delta s \to 0} \frac{\oint \vec{H} \cdot d\vec{l}}{\Delta s}$$

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(7) What is the significance of divergence and curl.

Divergence of a field gives net outflow of field per unit volume

Curl of a field gives the rate of rotation.

(8) State Helmholtz's theorem.

Helmholtz's theorem states that a general vector function F can be written as the sum of the gradient of a scalar function and the curl of the vector function.

$$\vec{F} = -\nabla V + \nabla \times \vec{A}.$$

(9) Explain the term irrotational & solenoidal field.

The vector field having its curl zero is called irrotational field.

$$\nabla \times \vec{A} = 0$$

The vector field having its divergence zero is called solenoidal field

$$\nabla \cdot \vec{A} = 0$$

(10) Write the line, surface and volume integrals.

Line integral $\rightarrow \int_l \vec{A} \cdot d\vec{l}$

Surface integral $\rightarrow \oint_s \vec{A} \cdot d\vec{s}$

Volume integral $\rightarrow \oint_v \rho_v \, dv$

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Problems

① Convert the point P(3,4,5) from cartesian to spherical coordinates.

Given $x = 3$, $y = 4$ and $z = 5$

To find: r, θ, ϕ

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$$

$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos^{-1} \frac{5}{5\sqrt{2}} = \cos^{-1}(1/\sqrt{2}) = 45^\circ$$

$$\phi = \tan^{-1}(y/x) = \tan^{-1} 4/3 = 53.13^\circ$$

∴ Spherical coordinate point is $P(5\sqrt{2}, 45^\circ, 53.13^\circ)$

② Convert the given point $P(2, \pi/2, \pi/3)$ in spherical coordinates into cartesian coordinates.

Given: $r = 2$, $\theta = \pi/2$ $\phi = \pi/3$

$$x = r \sin\theta \cos\phi = 2 \sin \pi/2 . \cos \pi/3 = 1$$

$$y = r \sin\theta \sin\phi = 2 \sin \pi/2 \sin \pi/3 = \sqrt{3}$$

$$z = r \cos\theta = 2 \cos \pi/2 = 0$$

∴ Cartesian coordinate point is $P(1, \sqrt{3}, 0)$.

③ Transform the cartesian coordinate (2,1,3) into cylindrical coordinate

Given: $x = 2$, $y = 1$ and $z = 3$

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$r = \sqrt{x^2+y^2} = \sqrt{2^2+1^2} = \sqrt{5} = 2.236$

$\phi = \tan^{-1} y/x = \tan^{-1}(1/2) = 26.56^\circ$

$z = z = 3$

$\therefore$ cylindrical coordinate point is $P(2.236, 26.56^\circ, 3)$

(4) Find the gradient of Scalar System $V = x^2 y + e^z$ at point $P(1, 5, -2)$.

Given: Cartesian c-s, $V = x^2 y + e^z$

$$\nabla V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

$$= \frac{\partial}{\partial x}(x^2 y + e^z)\vec{a_x} + \frac{\partial}{\partial y}(x^2 y + e^z)\vec{a_y} + \frac{\partial}{\partial z}(x^2 y + e^z)\vec{a_z}$$

$$= 2xy\,\vec{a_x} + x^2\,\vec{a_y} + e^z\,\vec{a_z}$$

∇V at $P(1, 5, -2)$ is

$$\nabla V = 2\times1\times5\,\vec{a_x} + (1)^2\,\vec{a_y} + e^{-2}\,\vec{a_z}$$

$$\boxed{\nabla V = 10\vec{a_x} + \vec{a_y} + 0.13\,\vec{a_z}}$$

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(5) Given $\vec{A} = (y \cos ax)\vec{a_x} + (y + e^x)\vec{a_z}$

Find $\nabla \times \vec{A}$ at the origin.

In cartesian coordinates

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y\cos ax & 0 & y+e^x \end{vmatrix}$$

$$= \frac{\partial}{\partial y}(y+e^x)\,\vec{a_x} - \left(\frac{\partial}{\partial x}(y+e^x) - \frac{\partial}{\partial z}(y\cos ax)\right)\vec{a_y}$$

$$- \frac{\partial}{\partial y}(y\cos ax)\,\vec{a_z}$$

$$\nabla \times \vec{A} = \vec{a_x} - e^x\vec{a_y} - \cos ax\,\vec{a_z}$$

$\nabla \times \vec{A}$ at origin i.e, $x = y = z = 0$

$$\therefore \nabla \times \vec{A} = \vec{a_x} - \vec{a_y} - \vec{a_z}.$$

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(6) Determine the divergence and curl of the given field $\vec{A} = 30\vec{a_x} + 2xy\,\vec{a_y} + 5xz^2\,\vec{a_z}$ at $(1, 1, -0.2)$ and hence state the nature of the field.

Solution: Given: $A_x = 30$, $A_y = 2xy$ and $A_z = 5xz^2$

To find $\nabla \cdot \vec{A}$ in cartesian coordinates

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$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(30) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(5xz^2)$$

$$= 0 + 2x + 10xz$$

At point $P(1, 1, -0.2)$

$$\nabla \cdot \vec{A} = 2 \times 1 + 10 \times 1 \times (-0.2) = 0$$

$\therefore \nabla \cdot \vec{A} = 0$, Hence field $\vec{A}$ is <u>solenoidal</u>

<u>To find $\nabla \times \vec{A}$</u>

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 30 & 2xy & 5xz^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(5xz^2) - \frac{\partial}{\partial z}(2xy)\right]\vec{a_x} - \left[\frac{\partial}{\partial x}(5xz^2) - \frac{\partial}{\partial z}(30)\right]\vec{a_y}$$

$$+ \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(30)\right]\vec{a_z}$$

$$= (0-0)\vec{a_x} - (5z^2 - 0)\vec{a_y} + 2y\,\vec{a_z}$$

$$= -5z^2\,\vec{a_y} + 2y\,\vec{a_z}$$

At point $P(1, 1, -0.2)$

$$\nabla \times \vec{A} = -5 \times (-0.2)\vec{a_y} + 2 \times 1\,\vec{a_z}$$

$$= 0.2\,\vec{a_y} + 2\,\vec{a_z} \neq 0$$

$\therefore$ Hence $\vec{A}$ is rotationtional field.

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(7) Given that $\vec{D} = \frac{5r^2}{4}\vec{a_r}$ C/m^2. Evaluate both sides of divergence theorem for the volume enclosed by $r = 4m$ and $\theta = \pi/4$.

Given: $\vec{D} = \frac{5r^2}{4}\vec{a_r}$ $\therefore$ $D_r = \frac{5r^2}{4}$

According to divergence theorem:

$$\oint_s \vec{D}\cdot\vec{ds} = \oint_v \nabla\cdot\vec{D}\, dv \quad \rightarrow (1)$$

To find L.H.S of eq (1), using spherical c-s

$\vec{ds_r} = r^2 \sin\theta\, d\theta\, d\phi\, \vec{a_r}$, $0 < \theta < \pi/4$

and $0 \le \phi \le 2\pi$

$$\therefore \oint_s \vec{D}\cdot\vec{ds} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\left(\frac{5r^2}{4}\vec{a_r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\, \vec{a_r}\right)$$

But $\therefore \vec{a_r}\cdot\vec{a_r} = 1$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\frac{5r^2}{4}\cdot r^2\sin\theta\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\frac{5r^4}{4}\sin\theta\, d\theta\, d\phi$$

$$= \frac{5r^4}{4}\cdot\left(-\cos\theta\right)_0^{\pi/4}\cdot(\phi)_0^{2\pi}$$

$$= \frac{5}{4}(4)^4 \times \left(-\cos\pi/4\right)(2\pi) = 588.89$$

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To find RHS of eq ①.

$$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \; D_\theta)$$

$$+ \frac{1}{r\sin\theta}\frac{\partial D\phi}{\partial \phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{5r^2}{4}\right) + 0 + 0$$

$$= \frac{5}{4r^2}(4r^3) = 5r$$

$$\therefore \oint_v \nabla \cdot \vec{D}\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{r=0}^{4}(5r)\,(r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$$\therefore dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$\oint_v \nabla \cdot \vec{D}\, dv = 5\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/4}\int_{r=0}^{4} r^3 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 5\left(\frac{r^4}{4}\right)_0^4 (-\cos\theta)_0^{\pi/4} (\phi)_0^{2\pi}$$

$$= \frac{5}{4}(4^4)\,(-\cos\pi/4 + \cos 0)\,(2\pi)$$

$$\oint_v \nabla \cdot \vec{D}\, dv = 588.89$$

Hence, $\oint_s \vec{D} \cdot d\vec{s} = \oint_v \nabla \cdot \vec{D}\, dv$

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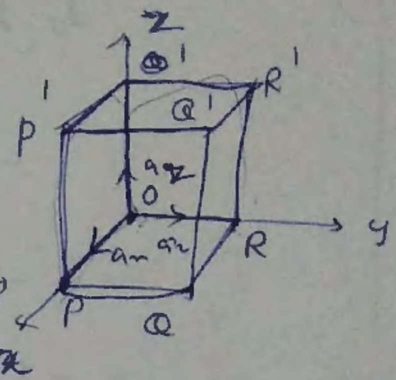
If $\vec{D} = 4xz\,\vec{a_x} - y^2\vec{a_y} + yz\,\vec{a_z}$, evaluate $\int_s \vec{D}\cdot\vec{ds}$ over the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$

Solution:



$$\int \vec{D}\cdot\vec{ds} = \int_1 \vec{D}\cdot\vec{ds} + \int_2 \vec{D}\cdot\vec{ds} + \int_3 \vec{D}\cdot\vec{ds} + \int_4 \vec{D}\cdot\vec{ds} + \int_5 \vec{D}\cdot ds + \int_6 \vec{D}\cdot\vec{ds}$$

i) For $\iint_{ORR'O'} \vec{D}\cdot\vec{ds}$, $x=0$, $\vec{ds} = dy\,dz(-\vec{a_x})$

$$= \int_0^1\int_0^1 (-y^2\vec{a_y} + yz\,\vec{a_z})\cdot(dy\,dz(-\vec{a_x}))$$

$$= 0$$

ii) For $\iint_{PQQ'P'} \vec{D}\cdot\vec{ds}$, $x=1$, $\vec{ds} = dy\,dz\,\vec{a_x}$

$$= \int_0^1\int_0^1 (4z\,\vec{a_x} - y^2\vec{a_y} + yz\,\vec{a_z})\cdot(dy\,dz\,\vec{a_x})$$

$$= 4\int_0^1\int_0^1 z\,dy\,dz \quad = 4\,(y)_0^1\left(\frac{z^2}{2}\right)_0^1$$

$$= 4\times 1\times\frac{1}{2}$$

$$= 2.$$

iii) For $\iint_{POO'P'} \vec{D}\cdot\vec{ds}$, $y=0$, $\vec{ds} = dx\,dz(-\vec{a_y})$

$$= \int_0^1\int_0^1 (4xz\,\vec{a_x} + yz\,\vec{a_z})\cdot(dx\,dz(-\vec{a_y}))$$

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iv) For $\iint_{QRR'Q'} \vec{D}\cdot\vec{ds}$, $y=1$, $\vec{ds} = dx\,dz\,\vec{a_y}$

$$= \int_0^1\int_0^1 (4xz\,\vec{a_x} - \vec{a_y} + z\,\vec{a_z})\cdot(dx\,dz\,\vec{a_y})$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -(x)_0^1\,(z)_0^1 = -1\times 1$$

$$= -1$$

v) For $\iint_{OPQR} \vec{D}\cdot\vec{ds}$, $z=0$, $\vec{ds} = dx\,dy\,(-\vec{a_z})$

$$= \int_0^1\int_0^1 (-y^2\,\vec{a_y})\cdot(dx\,dy\,(-\vec{a_z}))$$

$$= 0$$

vi) For $\iint_{O'P'Q'R'} \vec{D}\cdot\vec{ds}$, $z=1$, $\vec{ds} = dx\,dy\,\vec{a_z}$

$$= \int_0^1\int_0^1 (4x\,\vec{a_x} - y^2\,\vec{a_y} + y\,\vec{a_z})\cdot(dx\,dy\,\vec{a_z})$$

$$= \int_0^1\int_0^1 y\,dx\,dy$$

$$= (x)_0^1\left(\frac{y^2}{2}\right)_0^1$$

$$= 1\times\frac{1}{2} = 1/2$$

$\therefore \int_s \vec{D}\cdot\vec{ds} = 0+2+0-1+0+1/2 = 3/2$

$$\boxed{\oint_S \vec{D}\cdot\vec{ds} = 3/2}$$

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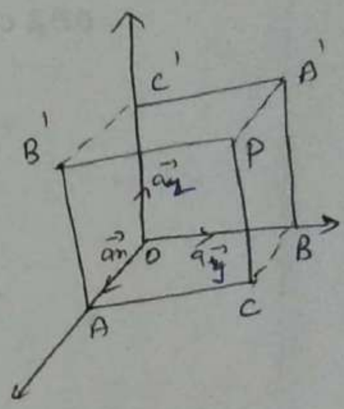
Check validity of the divergence theorem considering (33)
the field $\vec{D} = 2xy\,\vec{a_x} + x^2\,\vec{a_y}$ c/m^2 and the
rectangular parallelopiped formed by the planes $x=0,\ x=1,$
$y=0,\ y=2$ and $z=0,\ z=3$.

By divergence theorem.

$$\oint_S \vec{D}\cdot d\vec{s} = \oint_V \nabla\cdot\vec{D}\,dv \rightarrow (1)$$



To find R.H.S of eq (1)

$$\nabla\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2)$$

$$= 2y$$

$$\therefore \oint_V \nabla\cdot\vec{D}\,dv = \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} 2y\,dx\,dy\,dz$$

$$= (x)_0^1\, 2\times\left(\frac{y^2}{2}\right)_0^2 (z)_0^3$$

$$= 1\times\cancel{2}\,\frac{4}{\cancel{2}}\times 3 = 12$$

To find L.H.S eq (1),

(i) For $\iint_{ACPB'} \vec{D}\cdot(dy\,dz\,\vec{a_x})$, $x=1$ $\quad d\vec{s} = dy\,dz\,\vec{a_x}$

$$= \int_{y=0}^{2}\int_{z=0}^{3} (2y\,\vec{a_x} + \vec{a_y})\cdot(dy\,dz\,\vec{a_x})$$

$$= \int_0^2\int_0^3 2y\,dy\,dz = 2\left(\frac{y^2}{2}\right)_0^2 (z)_0^3$$

$$= 2\times\frac{4}{2}\times 3 = 12$$

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ii) For $\iint_{OBA'C'} \vec{D} \cdot d\vec{s}$, $x = 0$, $d\vec{s} = dy\,dz(-\vec{a_x})$

$$= \int_{y=0}^{2}\int_{z=0}^{3} (0) \cdot dy\,dz(-\vec{a_x}) = 0$$

iii) For $\iint_{OAB'C'} \vec{D} \cdot d\vec{s}$, $y = 0$. $d\vec{s} = dx\,dz\,(-a_y)$

$$= \int_{x=0}^{1}\int_{z=0}^{3} (+x^2\vec{a_y}) \cdot dx\,dz(-\vec{a_y})$$

$$= -\left(\frac{x^3}{3}\right)_0^1 (z)_0^3 =$$

$$= -\frac{1}{3} \times 3 = -1$$

iv) For $\iint_{PA'BC} \vec{D} \cdot d\vec{s}$, $y = 2$, $d\vec{s} = dx\,dz(\vec{a_y})$

$$= \int_{x=0}^{1}\int_{z=0}^{3} (4x\,\vec{a_x} + x^2\vec{a_y}) \cdot (dx\,dz\,\vec{a_y})$$

$$= \int_{x=0}^{1}\int_{z=0}^{3} x^2\,dx\,dz$$

$$= \left(\frac{x^3}{3}\right)_0^1 (z)_0^3$$

$$= 1$$

v) For $\iint_{OACB} \vec{D} \cdot d\vec{s}$, $d\vec{s} = dx\,dy\,(-\vec{a_z})$

$z = 0$

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$$= \int_{x=0}^{1} \int_{y=0}^{2} (2xy\,\vec{a_x} + x^2\,\vec{a_y})\,(dx\,dy\,(-\vec{a_z}))$$ (35)

$= 0$

vi) for $\iint_{PA'C'B'} \vec{D} \cdot \vec{ds}$, $z = 3$, $\vec{ds} = dx\,dy\,\vec{a_z}$

$$= \int_{x=0}^{1} \int_{y=0}^{2} (2xy\,\vec{a_x} + x^2\,\vec{a_y}) \cdot (dx\,dy\,\vec{a_z})$$

$= 0$.

$\therefore \oint_S \vec{D} \cdot \vec{ds} = 12 + 0 + 1 - 1 + 0 + 0 = 12.$

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1. Determine the divergence and curl of the given field $F = 30\bar{a}_x + 2xy\bar{a}_y + 5xz^2\bar{a}_z$ at $(1,1,-0.2)$ and hence state the nature of the field. [Nov/Dec'2010, AU, ECE]

Soln:-

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= \frac{\partial}{\partial x}(30) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(5xz^2)$$

$$= 0 + 2x + 10xz$$

$$\nabla \cdot F = 2x + 10xz$$

At $(1,1,-0.2)$

$$\nabla \cdot F = 2(1) + 10(1)(0.2) = 2 - 2 = 0$$

Since divergence is zero, it is solenoidal.

Curl:
$$\nabla \times F = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 30 & 2xy & 5xz^2 \end{vmatrix}$$

$$= \bar{a}_x[0-0] - \bar{a}_y[5z^2 - 0] + \bar{a}_z[2y - 0]$$

$$\nabla \times F = -5z^2\bar{a}_y + 2y\bar{a}_z$$

At $(1,1,-0.2)$

$$\nabla \times F = -5(-0.2)^2\bar{a}_y + 2(1)\bar{a}_z$$

$$\nabla \times F = -0.2\bar{a}_y + 2\bar{a}_z$$

$$|\nabla \times F| = \sqrt{(-0.2)^2 + 2^2} = \sqrt{4.04}$$

Since curl is not a non zero, it is ~~solenoidal~~ not irrotational field.

2. Determine the divergence of these vector fields. AU ECE (Nov/Dec'2007)

(i) $P = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

(ii) $Q = \rho\sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

(iii) $T = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

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Soln: (i) $P = x^2yz\,\hat{a}_x + xz\,\hat{a}_z$

$\nabla \cdot P = \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$

$\nabla \cdot P = 2xyz + x$

(ii) $Q = \rho \sin\phi\,\hat{a}_\rho + \rho^2 z\,\hat{a}_\phi + z\cos\phi\,\hat{a}_z$

$\nabla \cdot Q = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_z}{\partial z}$

$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$

$= \frac{1}{\rho}2\rho\sin\phi + \frac{1}{\rho}(0) + \cos\phi$

$= 2\sin\phi + \cos\phi$

(iii) $T = \frac{1}{r^2}\cos\theta\,\hat{a}_r + r\sin\theta\cos\phi\,\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

$\nabla \cdot T = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, T_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\cos\theta}{r^2}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\cos\theta)$

$= 2\cos\theta\cos\phi$

(3) Given $A = (y\cos ax)\,\bar{a}_x + (y + e^x)\,\bar{a}_z$. Find $\nabla \times A$ at the origin. (AU Nov/Dec 2009)

Soln:
$$\nabla \times A = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\cos ax & 0 & y + e^x \end{vmatrix}$$

$= \bar{a}_x\left[\frac{\partial}{\partial y}(y + e^x) - 0\right] - \bar{a}_y\left[\frac{\partial}{\partial x}(y + e^x) - \frac{\partial}{\partial z}(y\cos ax)\right]$
$+ \bar{a}_z\left[-\frac{\partial}{\partial y}(y\cos ax)\right]$

$= \bar{a}_x - e^x\,\bar{a}_y - \cos ax\,\bar{a}_z$

(4) If two vectors are expressed in cylindrical coordinates as

$A = 2\bar{a}_\rho + \pi\,\bar{a}_\phi + \bar{a}_z$

$B = -\bar{a}_\rho + \frac{3\pi}{2}\bar{a}_\phi - 2\bar{a}_z$

Compute a unit vector perpendicular to the plane containing A and B [AU May/June 2009]

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Soln:-

$$\bar{A}\times\bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & \pi & 1 \\ -1 & \frac{3\pi}{2} & -2 \end{vmatrix}$$

$$= \hat{a}_x\left(-2\pi - \frac{3\pi}{2}\right) - \hat{a}_y(-4+1) + \hat{a}_z(3\pi+\pi)$$

$$= \hat{a}_x\left(-\frac{7\pi}{2}\right) + 3\hat{a}_y + 4\pi\,\hat{a}_z$$

$$= -10.99\,\hat{a}_x + 3\,\hat{a}_y + 12.5\,\hat{a}_z$$

$$|\bar{A}\times\bar{B}| = \sqrt{(-10.99)^2 + 3^2 + 12.5^2} = 17.$$

$$\hat{a}_n = \frac{\bar{A}\times\bar{B}}{|\bar{A}\times\bar{B}|} = \frac{-10.99\,\bar{a}_x + 3\,\bar{a}_y + 12.5\,\bar{a}_z}{17}$$

$$= -0.646\,\bar{a}_x + 0.176\,\bar{a}_y + 0.735\,\bar{a}_z$$

(5) Use spherical co-ordinates and integrate to find the area of the region $0 \le \phi \le \alpha$ on the spherical shell of radius a. What is the area of $\alpha = 2\pi$? AU, ECE (Nov / Dec 2007)

Soln:-

The differential surface area when the radius is constant, in spherical co-ordinates are $r^2 \sin\theta\, d\theta\, d\phi$

The total area of sphere with $r = a$.

$$\int_{\phi=0}^{\alpha}\int_{\theta=0}^{\pi} a^2 \sin\theta\, d\theta\, d\phi = a^2\int_{\theta=0}^{\pi}\sin\theta\, d\theta \int_{\phi=0}^{\alpha} d\phi$$

$$= a^2\left[-\cos\theta\right]_0^{\pi} \int_{\phi=0}^{\alpha} d\phi$$

$$= a^2\left[-\cos\pi + \cos 0\right]\left[\phi\right]_0^{\alpha}$$

$$= a^2\left[(-1)+1\right]\left[\alpha\right]$$

$$= 2\alpha a^2$$

If $\alpha = 2\pi$, then Area $= 2\alpha a^2 = 2(2\pi)a^2 = 4\pi a^2$

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⑥ Given the two points

A (x = 2, y = 3, z = -1) and B (r = 4, θ = 25°, φ = 120°)

Find the spherical co-ordinates of A and cartesian co-ordinates of B

[AU, ECE, April / May 2010]

Soln:- Given A (x = 2, y = 3, z = -1)

The corresponding spherical co-ordinates are

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+(-1)^2} = \sqrt{14} = 3.74$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left[\frac{(-1)}{3.74}\right] = \cos^{-1}(-0.267)$$

$$\theta = 105.5^\circ$$

$$\phi = \tan^{-1} y/x = \tan^{-1}(3/2) = \tan^{-1} 1.5$$

$$\phi = 56.31^\circ$$

Given B [r = 4, θ = 25°, φ = 120°]

The corresponding cartesian co-ordinates are

$$x = r\sin\theta\cos\phi = 4\sin 25^\circ \cos 120^\circ = 4(0.42)(-0.5)$$

$$x = -0.845$$

$$y = r\sin\theta\sin\phi = 4\sin 25^\circ \sin 120^\circ$$

$$y = 1.464$$

$$z = r\cos\theta = 4\cos 25^\circ$$

$$z = 3.625$$

⑦ Find curl $\bar{H}$, if $\bar{H} = (2\rho\cos\phi\,\bar{a}_\rho - 4\rho\sin\phi\,\bar{a}_\phi + 3\bar{a}_z)$

[AU, ECE, April / May 2010]

Soln:-

$$\nabla \times \bar{H} = \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ H_\rho & \rho H_\phi & H_z \end{vmatrix}$$

$$= \frac{1}{\rho}\begin{vmatrix} \bar{a}_\rho & \rho\bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 2\rho\cos\phi & -4\rho^2\sin\phi & 3 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[\frac{\partial}{\partial\phi}(3) - \frac{\partial}{\partial z}(-4\rho^2\sin\phi)\right]\bar{a}_\rho + \left[\frac{\partial}{\partial z}(2\rho\cos\phi) - \frac{\partial}{\partial\rho}(3)\right]\bar{a}_\rho$$

$$+ \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(-4\rho^2\sin\phi) - \frac{\partial}{\partial\phi}(2\rho\cos\phi)\right]\bar{a}_z$$

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$$= \frac{1}{\rho}(0)a_\rho + 0a_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(-4\rho^2\sin\phi) - \frac{\partial}{\partial\phi}(2\rho\cos\phi)\right]\bar{a_z}$$

$$= \frac{1}{\rho}\left[-8\rho\sin\phi + 2\rho\sin\phi\right]\bar{a_z}$$

$$= \left[-8\sin\phi + 2\sin\phi\right]\bar{a_z}$$

$$= -6\sin\phi\,\bar{a_z}$$

) Using divergence theorem, evaluate $\int_S \bar{F}.ds$ where $\bar{F} = 2xy\bar{i} + y^2\bar{j} + 4yz\bar{k}$ and S is the surface of the cube bounded by $x=0, x=1$; $y=0, y=1$ and $z=0, z=1$.

Soln:- By using divergence theorem

$$\iiint \text{Div}\bar{F}.dv = \iint \bar{F}.dS$$

$$\text{Div}\bar{F} = \nabla\cdot\bar{F} = \left(\frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}\right).(2xy\bar{i} + y^2\bar{j} + 4yz\bar{k})$$

$$= \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(4yz)$$

$$= 2y + 2y + 4y = 8y$$

$$\iiint \text{Div}\bar{F}.dv = \int_0^1\int_0^1\int_0^1 8y\,dx\,dy\,dz$$

$$= \int_0^1\left[8\frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^1 4dz = 4$$

(9) Evaluate both sides of divergence theorem for the field $D = \frac{10y^3}{3}\hat{a_y}$, for the cube 2 m on each edge centred at the origin and with the edges parallel to the axes. (AU, EEE, Nov/Dec' 2007)

Solution:- Divergence theorem is

$$\int_S D.ds = \int_V \nabla.D\,dv$$

$$\text{L.H.S} = \iint D.ds = \iint D_{y=-1}\,dz\,dx(-a_y) + \iint D_{y=1}\,dz\,dx\,a_y$$

Since the cube is placed at the centre of the co-ordinate system, the limits are taken as

$-1 < x < 1$, $-1 < y < 1$ and $-1 < z < 1$

$$\iint D.ds = \int_{x=-1}^{1}\int_{z=-1}^{1}\frac{10(-1)^3}{3}dz\,dx\,a_y(-a_y) + \int_{x=-1}^{1}\int_{z=-1}^{1}\frac{10(1)^3}{3}dz\,dx\,a_y.a_y$$

$$= \frac{10}{3}[z]_{-1}^{1}[x]_{-1}^{1} + \frac{10}{3}[z]_{-1}^{1}[x]_{-1}^{1}$$

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$$= \frac{10}{3}\left[1-(-1)\right]\left[1-(-1)\right] + \frac{10}{3}\left[1-\ldots\right]$$

$$= \frac{10}{3}(2)(2) + \frac{10}{3}(2)(2)$$

$$\text{L.H.S} = \frac{40}{3} + \frac{40}{3} = \frac{80}{3}$$

$$\text{R.H.S} = \iiint \nabla\cdot D\, dv = \int_{x=-1}^{1}\int_{y=-1}^{1}\int_{z=-1}^{1} 10y^2\, dx\, dy\, dz$$

$$= \frac{10}{3}\left[\frac{y^3}{3}\right]_{-1}^{1}\left[x\right]_{-1}^{1}\left[y\right]_{-1}^{1}$$

$$= \frac{10}{3}\left[1^3-(-1)^3\right]\left[1-(-1)\right]\left[1-(-1)\right]$$

$$= \frac{10}{3}(2)(2)(2)$$

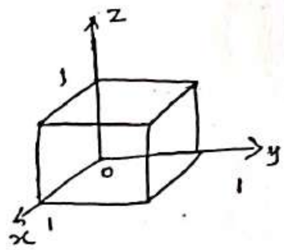
$$\text{R.H.S} = \frac{80}{3}$$

$$\text{L.H.S} = \text{R.H.S}$$

Hence divergence theorem is verified.

(10) Verify the divergence theorem for the following case $\bar{A} = xy^2\bar{a_x} + y^3\bar{a_y} + y^2z\bar{a_z}$ and the surface is a cuboid defined by $0<x<1$, $0<y<1$ and $0<z<1$

Soln:- $\iint A\cdot ds = \iiint \nabla\cdot A\, dv$



L.H.S A cuboid has six surfaces

$$= \iint A\cdot ds = \iint A_{x=0}\, dy\,dz + \iint A_{x=1}\, dy\,dz$$
$$+ \iint A_{y=0}\, dx\,dz + \iint A_{y=1}\, dx\,dz$$
$$+ \iint A_{z=0}\, dx\,dy + \iint A_{z=1}\, dx\,dy$$

$$\iint A\cdot ds = \int_{z=0}^{1}\int_{y=0}^{1} y^2\, dy\,dz + \int_{z=0}^{1}\int_{x=0}^{1} dx\,dz + \int_{y=0}^{1}\int_{x=0}^{1} y^2\, dx\,dy$$

$$= \left[\frac{y^3}{3}\right]_0^1\left[z\right]_0^1 + \left[x\right]_0^1\left[z\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1\left[x\right]_0^1$$

$$= \frac{1}{3} + 1 + \frac{1}{3} = \frac{2}{3} + 1 = \frac{2+3}{3} = 5/3$$

$$\text{L.H.S} = 5/3$$

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R.H.S $= \iiint \nabla \cdot A\, dv$

$\nabla \cdot A = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(y^2 z)$

$= y^2 + 3y^2 + y^2 = 5y^2$

$$\int \nabla \cdot A\, dv = \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{1} 5y^2\, dx\, dy\, dz$$

$$= 5\left[\frac{y^3}{3}\right]_0^1 [x]_0^1 [z]_0^1$$

R.H.S $= \frac{5}{3}$

L.H.S = R.H.S

Thus divergence theorem is verified.

11) Given that $\bar{D} = \left(\frac{10\rho^3}{4}\right)\bar{a}_\rho$ in cylindrical co-ordinates, evaluate both sides of divergence theorem for the volume enclosed by $\rho = 4, z=0$ and $z = 5$.

Soln:- $\iint D \cdot ds = \iiint \nabla \cdot D\, dv$

$$\text{L.H.S} = \iint D \cdot ds = \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \left(\frac{10\rho^3}{4}\right)_{\rho=4} a_\rho \cdot \rho\, d\phi\, dz\, a_\rho$$

$$= \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \frac{10 \times 4^3 \times 4}{4}\, d\phi\, dz$$

$$= 640 \int_{z=0}^{5} \int_{\phi=0}^{2\pi} d\phi\, dz = 640\, [1]_0^5\, [1]_0^{2\pi}$$

$= 640 \times 5 \times 2\pi$

L.H.S $= 6400\pi$

$$\text{R.H.S} = \iiint \nabla \cdot D\, dv = \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{4} 10\rho^3\, d\rho\, d\phi\, dz$$

$$\cancel{\left(= 10\left[\frac{\rho^4}{4}\right]_{\rho=0}^{4} \int_{z=0}^{5} \int_{\phi=0}^{2\pi} d\phi\, dz\right)}$$

$$\nabla \cdot D = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho)$$

$$\nabla \cdot D = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{10\rho^4}{4}\right) = \frac{1}{\rho}\, \frac{4 \cdot 10\rho^3}{4} = 10\rho^3$$

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$$\iiint \nabla\cdot D\, dv = \int_{z=0} \int_{\phi=0} \int_{\rho=0} 10\rho^{2} \rho\, d\rho\, d\phi\, dz$$

$$= 10\left[\frac{\rho^4}{4}\right]_{\rho=0}^{4} \int_{z=0}^{5}\int_{\phi=0}^{2\pi} d\phi\, dz$$

$$= 640 \times 2\pi \times 5$$

$$R.H.S = 6400\pi$$

$$L.H.S = R.H.S$$

Hence, divergence theorem is verified.

(12) A Vector field $D = \frac{5r^2}{4} I_r$ is given in spherical co-ordinates. Evaluate both sides of divergence theorem for the volume enclos between $r=1$ and $r=2$.

Soln:- Divergence theorem

$$\iint D\cdot ds = \iiint \nabla\cdot D dv$$

$$L.H.S = \iint D\cdot ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{5r^2}{4}\right)_{r=1} I_r\cdot(r^2\sin\theta\, d\theta\, d\phi)(-I_r)$$

$$+ \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\frac{5r^2}{4}\right)_{r=2} I_r\cdot[r^2\sin\theta\, d\theta\, d\phi](I_r)$$

$$= \int_0^{2\pi}\int_0^{\pi} -\frac{5}{4}\sin\theta\, d\theta\, d\phi + \int_0^{2\pi}\int_0^{\pi} 5\times 4\sin\theta\, d\theta\, d\phi$$

$$= -\frac{5}{4}\int_0^{2\pi}(-\cos\theta)_0^{\pi} d\phi + 20\int_0^{2\pi}(-\cos\theta)_0^{\pi} d\phi$$

$$= \frac{5}{4}(-2)[\phi]_0^{2\pi} + 20(2)[\phi]_0^{2\pi}$$

$$= -\frac{5}{2}(2\pi) + 40\times 2\pi$$

$$= 80\pi - 5\pi$$

$$LHS = \iint D\cdot dS = 75\pi$$

$$R.H.S = \iiint \nabla\cdot D dv$$

$$\nabla\cdot D = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{5}{4}r^2\right]$$

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$$= \frac{1}{r^2} \frac{4 \times 5 r^3}{4}$$

$$= \frac{1}{r^2} 5r^3$$

$$\nabla \cdot D = 5r$$

$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$\iiint \nabla \cdot D \, dv = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=1}^{2} 5r \times r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_1^{2} 5r^3 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 5 \int_0^{2\pi} \int_0^{\pi} \left(\frac{r^4}{4}\right)_1^2 \sin\theta \, d\theta \, d\phi$$

$$= 5 \times \left[\frac{16}{4} - \frac{1}{4}\right] \int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi$$

$$= 5 \times \frac{15}{4} \int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= 5 \times \frac{15}{4} (2) \int_0^{2\pi} d\phi$$

$$= \frac{75}{2} \left[\phi\right]_0^{2\pi}$$

$$\iiint \nabla \cdot D \, dv = 75\pi = R.H.S$$

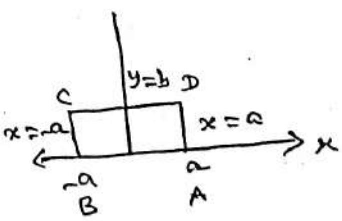
L.H.S = R.H.S

Hence Divergence theorem is verified.

(13) Given that $\bar{A} = (x^2+y^2)\,\bar{a_x} - 2xy\,\bar{a_y}$, evaluate both side of stoke's theorem for a rectangular path bounded by the line $x = \pm a$, $y = 0$, $y = b$.

soln:- Stoke's theorem

$$\int_L A \cdot dl = \iint (\nabla \times A) \cdot ds$$



$$L.H.S = \int_L \bar{A} \cdot \bar{dl}$$

Consider $\bar{A} \cdot \bar{dl} = \left[(x^2+y^2)\,\hat{a_x} - 2xy\,\hat{a_y}\right] \cdot \left[dx\,\hat{a_x} + dy\,\hat{a_y} + dz\,\hat{a_z}\right]$

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$\bar{A} \cdot d\bar{l} = (x^2+y^2)\,dx - (2xy)\,dy$

$\int_A^D \bar{A} \cdot d\bar{l}$ = Along AD

$x = a, \quad y = 0, b$

$\Rightarrow dx = 0$

$\bar{A} \cdot d\bar{l} = \left[(a^2+y^2)\,\overset{0}{\overset{\shortparallel}{da}} + (-2ay)\,dy\right]$

$\int \bar{A} \cdot d\bar{l} = \int_0^b (a^2+y^2)\,0 + (-2ay)\,dy$

$= \left[-2a\,\frac{y^2}{2}\right]_0^b$

$\boxed{\int_{AD} \bar{A} \cdot d\bar{l} = -ab^2}$

Along DC : $\int_{DC} \bar{A} \cdot d\bar{l} \Rightarrow y = b, \; x = +a$ to $-a$

$\Rightarrow \int_{+a}^{-a} x^2\,dx + b^2\,dx - 2xb\,\overset{=0}{(dy)}$

$= \left[\frac{x^3}{3} + b^2 x\right]_{+a}^{-a}$

$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2}{3}a^3 - 2ab^2$

$= -\left(\frac{2}{3}a^3 + 2ab^2\right)$

$\boxed{\int_{DC} \bar{A} \cdot d\bar{l} = -\frac{2}{3}\left[a^3 + 3ab^2\right]}$

Along CB : $\int_{CB} \bar{A} \cdot d\bar{l}$

Here $x = -a, \quad y = b$ to 0

$dx = 0$

$\int_b^0 \left[(a^2+y^2)\,0 + 2ay\,dy\right] = \int_b^0 2ay\,dy = \left[\frac{2ay^2}{2}\right]_b^0 = 0 - ab^2$

$\boxed{\int_{CB} \bar{A} \cdot d\bar{l} = -ab^2}$

Along BA = $\int_{BA} \bar{A} \cdot d\bar{l}$; Here $y = 0, \; x = -a$ to a

$\int_{-a}^{+a} x^2\,dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = 2\frac{a^3}{3}$

$\boxed{\int \bar{A} \cdot d\bar{l} = \frac{2a^3}{3}}$

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$$\therefore \int \bar{A}\cdot d\bar{l} = \int_{AD} \bar{A}\cdot d\bar{l} + \int_{DC} \bar{A}\cdot d\bar{l} + \int_{CB} \bar{A}\cdot d\bar{l} + \int_{BA} \bar{A}\cdot d\bar{l}$$

$$\int \bar{A}\cdot d\bar{l} = -ab^2 - \frac{2}{3}a^3 - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$$

$$\boxed{\int \bar{A}\cdot d\bar{l} = -4ab^2} \quad \ldots (1)$$

RHS:- $\iint (\nabla \times \bar{A})\, d\bar{s}$

$$\nabla \times \bar{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \hat{a}_x\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right] - \hat{a}_y\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+y^2)\right] + \hat{a}_z\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right]$$

$$= \hat{a}_x[0-0] - \hat{a}_y[0] + \hat{a}_z[-2y-2y]$$

$$\boxed{\nabla \times \bar{A} = -4y\,\hat{a}_z}$$

Limits of x, $-a$ to a, Limit of y; $0 \to b$

$$\therefore \iint (\nabla \times \bar{A})\, ds = \int_{-a}^{+a}\int_0^b (-4y\,\hat{a}_z)\, dx\, dy$$

$$= \int_{-a}^{+a}\int_0^b -4y\, dx\, dy$$

$$= -\int_{-a}^{a}\left[\frac{4y^2}{2}\right]_0^b dx = -2\int_{-a}^{a} b^2\, dx$$

$$= -2\left[b^2 x\right]_{-a}^{a} = -2\left[ab^2 + ab^2\right] = -4ab^2$$

$$\boxed{\iint (\nabla \times \bar{A})\, ds = -4ab^2} \rightarrow (2)$$

From equation (1) and (2),

$$\oint_L A\cdot dl = \iint (\nabla \times \bar{A})\, ds$$

Hence proved.

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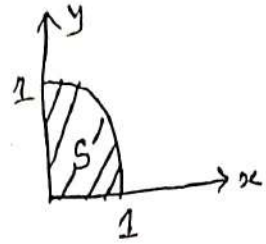
(14) Give $A = \rho\cos\phi\, a_\rho + \rho^2 a_z$ A. Compute $\nabla \times A$ and $\int_s \nabla \times A \cdot ds$ over the area S' as shown in the figure below.



(AU, EEE, Nov 2011)

Soln:-

$$\nabla \times A = \begin{vmatrix} \frac{a_\rho}{\rho} & a_\phi & \frac{a_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \begin{vmatrix} \frac{a_\rho}{\rho} & a_\phi & \frac{a_z}{\rho} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \rho\cos\phi & 0 & \rho^2 \end{vmatrix}$$

$$= \frac{a_\rho}{\rho}\left[\frac{\partial(\rho^2)}{\partial\phi}\right] - a_\phi\left[\frac{\partial}{\partial\rho}(\rho^2) - \frac{\partial}{\partial z}(\rho\cos\phi)\right] + \frac{a_z}{\rho}\left(\frac{\partial}{\partial\rho}(0) - \frac{\partial}{\partial\phi}(\rho\cos\phi)\right)$$

$$= -a_\phi(2\rho) + \frac{a_z}{\rho}(-\rho(-\sin\phi))$$

$$\nabla \times A = -a_\phi(2\rho) + a_z \sin\phi$$

$$ds = \rho\, d\rho\, d\phi\, a_z$$

$$\int_s (\nabla \times A) \cdot ds = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \sin\phi\, (\rho\, d\rho\, d\phi)$$

$$= \left[\frac{\rho^2}{2}\right]_0^1 \int_{\phi=0}^{\pi/2} \sin\phi\, d\phi$$

$$= \frac{1}{2}\left[-\cos\phi\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[-\cos\pi/2 + \cos 0\right]$$

$$\int_s (\nabla \times A)\, ds = 1/2$$

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Verify Stoke's theorem for a vector field

✓ $\bar{F} = r^2 \cos\phi \, \bar{a_r} + z \sin\phi \, \bar{a_z}$ around the path L defined by

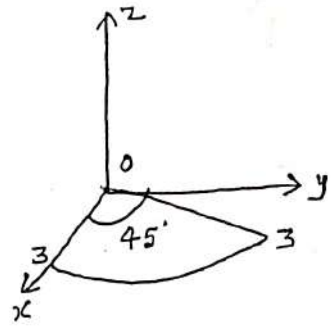
$0 \le r \le 3$, $0 \le \phi \le 45^\circ$ and $z = 0$. [AU, ECE, May/June 2009]

Soln:-

Stoke's theorem is

$$\oint F \cdot dl = \iint (\nabla \times F) \, ds$$

Divide path 'l' into 3 sections.



$$R.H.S = \iint (\nabla \times F) \, ds$$

Curl in cylindrical co-ordinate system

$$\nabla \times F = \frac{1}{r} \begin{vmatrix} \bar{a}_r & \bar{a}_\phi & \bar{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ r^2 \cos\phi & 0 & z\sin\phi \end{vmatrix}$$

Note:- Given axis are r, ϕ, z. So given system is in cylindrical co-ordinate system.

$$= \frac{1}{r} \left\{ \left[\bar{a}_r \frac{\partial}{\partial \phi}(z\sin\phi) - 0 \right] - \left[\bar{a}_\phi \left(\frac{\partial}{\partial r}(z\sin\phi) - \frac{\partial}{\partial z}(r^2\cos\phi) \right) \right] + \left[\bar{a}_z \left(0 - \frac{\partial}{\partial \phi}(r^2 \cos\phi) \right) \right] \right\}$$

$$= \frac{1}{r} \left\{ \bar{a_r}(z\cos\phi) - \bar{a_\phi}(0-0) + \bar{a_z}\left[-r^2(-\sin\phi) \right] \right\}$$

$$= \frac{1}{r} \left\{ z\cos\phi \, \bar{a_r} + r^2 \sin\phi \, \bar{a_z} \right\}$$

$$\nabla \times F = \frac{z}{r} \cos\phi \, \bar{a_r} + r \sin\phi \, \bar{a_z}$$

$$ds = r \, dr \, d\phi \, \bar{a_z}$$

Note
$\bar{a_r} \cdot \bar{a_r} = 1$
$\bar{a_r} \cdot \bar{a_z} = 0$

$$R.H.S = \iint (\nabla \times F) \cdot ds$$

$$= \int_{\phi=0}^{45^\circ} \int_{r=0}^{3} r \sin\phi \, r \, dr \, d\phi = \int_{\phi=0}^{45^\circ} \int_{r=0}^{3} r^2 \sin\phi \, dr \, d\phi$$

$$= \left[\frac{r^3}{3} \right]_0^3 \left[\cos\phi \right]_0^{45^\circ} = \frac{3^3}{3} \left[-\cos 45^\circ + \cos 0^\circ \right]$$

$$\boxed{R.H.S = 2.636}$$

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$L.H.S = \oint F.dl$

Divide path 'l' into 3 sections

Section - I :- r varies from 0 to 3, $\phi = 0$ and $z = 0$

$$\oint_{sec-I} F.dl = \int_0^3 (r^2 \cos\phi \, \bar{a_r} + z \sin\phi \, \bar{a_z}) . dr \, \bar{a_r} = \int_{r=0, \phi=0}^{3} r^2 \cos\phi \, dr$$

$$= \left[\frac{r^3}{3}\right]_0^3 = \frac{3^3}{3} = 9$$

Section - II $r = 3$, ϕ varies from 0 to 45°, $z = 0$, $dl = d\phi \, \bar{a_\phi}$

$$\oint_{sec-II} F.dl = \int_{z=0, \phi=0}^{45^\circ} (r^2 \cos\phi \, \bar{a_r} + z \sin\phi \, \bar{a_z}) . d\phi \, \bar{a_\phi}$$

note

$\bar{a_r} . \bar{a_r} = 1$

$\bar{a_r} . \bar{a_\phi} = 0$

$\bar{a_z} . \bar{a_\phi} = 0$

$$\oint_{sec-II} F.dl = 0$$

Section - III : r varies from 3 to 0, $\phi = 45$, $z = 0$, $dl = dr \, \bar{a_r}$

$$\oint_{sec-III} F.dl = \int_{r=3 \; \phi=45^\circ}^{0} (r^2 \cos\phi \, \bar{a_r} + z \sin\phi \, \bar{a_z}) . dr \, \bar{a_r}$$

$$= \int_{r=3 \; \phi=45}^{0} r^2 \cos\phi \, dr = \cos 45^\circ \int_{r=3}^{0} r^2 dr = \frac{1}{\sqrt{2}} \left[\frac{r^3}{3}\right]_3^0$$

$$= \frac{1}{\sqrt{2}} \left[0 - \frac{3^3}{3}\right] = \frac{1}{\sqrt{2}} (-9) = -6.364$$

$$L.H.S = \oint F.dl = \oint_{sec-I} F.dl + \oint_{sec-II} F.dl + \oint_{sec-III} F.dl$$

$$= 9 + 0 - 6.364$$

$$\boxed{L.H.S = 2.636}$$

$L.H.S = R.H.S$

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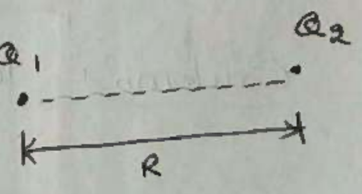
Electric field, coulomb's law, Gauss's law and applications, Electric potential, conductors in static electric field, Dielectrics in static electric field, Electric flux density and dielectric constant, Boundary conditions, capacitance, Parallel, cylindrical and spherical capacitors, Electrostatic energy, Poisson's and Laplace's equations, uniqueness of electrostatic solutions, current density and ohm's law, Electromotive force and kirchoff's voltage law, Equation of continuity and kirchoff's current law

* Coulomb's law :-

Statement :-

" The force between two point charges is proportional to the product of charges and is inversely proportional to the square of distance between them".



$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$\therefore F = k \cdot \frac{Q_1 Q_2}{R^2}$$

F = Force in Newton

Q_1, Q_2 = Product of two charges

R = distance between two charges

where

k = constant of proportionality

$$= \frac{1}{4\pi\varepsilon}$$

ε = permittivity of medium

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Unit of ε : $\frac{C^2}{N\text{-}m^2}$ (or) F/m   F - Farad

In general

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

$$\varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m} = \frac{10^{-9}}{36\pi} \text{ F/m}$$

For free space,

$$\varepsilon_r = 1$$

$$\therefore k = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^{9} \text{ m/F.}$$

Hence, Coulomb's law can be expressed as

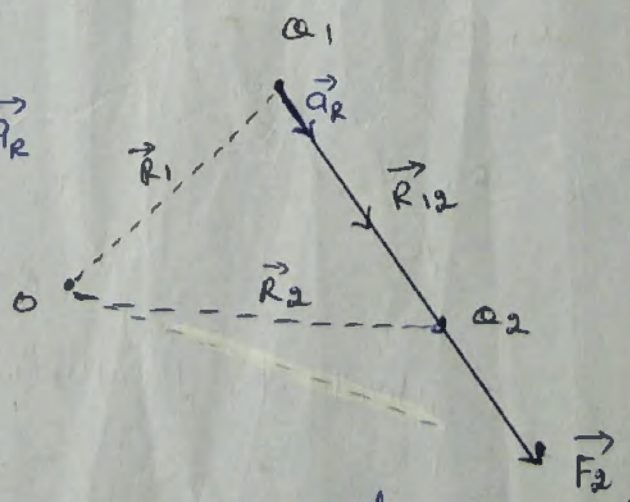
$$F = \frac{Q_1 Q_2}{4\pi\varepsilon R^2} \text{ in Newton (N)}$$

For free space,

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \text{ Newton}$$

* Coulomb's law in vector form :-

$$\vec{F} = k.\frac{Q_1 Q_2}{R^2}\vec{a}_R$$



vector form of Coulomb's law

* The force is a vector quantity
* Consider two point charges Q_1 and Q_2 having the same sign.

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According to Coulomb's law

i) the magnitude of force depends on the medium

ii) Like charges repel each other and unlike Charges attract each other

iii) Force is directed along the line joining the two charges.

$$\therefore \vec{F_2} = k \cdot \frac{Q_1 Q_2}{R_{12}^2} \vec{a}_{R_{12}} \text{ (Newton)} \Rightarrow \text{Force on charge } Q_2 \text{ due to } Q_1$$

where $\vec{a_R} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|}$; $k = \frac{1}{4\pi\varepsilon}$

$$\vec{R_{12}} = \vec{R_2} - \vec{R_1} \quad ; \quad |\vec{R_{12}}| = |\vec{R_2} - \vec{R_1}|$$

III^{ly}, Force on charge Q_1 due to Q_2

$$\vec{F_1} = k \frac{Q_1 Q_2}{R_{21}^2} \vec{a}_{R_{21}} \text{ (Newton)}$$

* Coulomb's law is linear

$$\therefore n\vec{F_2} = -n\vec{F_1}$$

$\therefore$ Force due to 'n' charges on Q_1

$$\vec{F}_{Total} = \vec{F}_{21} + \vec{F}_{31} + \vec{F}_{41} + \dots \vec{F}_{n1}$$

---

* **Electric field :-**

Electric field is defined as the region where the forces act.

Electric field is a vector quantity

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Definition of Electric field Intensity ($\vec{E}$):-

Electric field intensity is defined as the force per unit charge.

It is also called as electric field strength.

$$\therefore \vec{E} = \frac{\vec{F}}{Q} = \frac{\vec{F_{12}}}{Q_2}$$

$\vec{F_{12}}$ = vector force acting on Q_2 by Q_1

According to Coulomb's law

$$\vec{F_{12}} = k . \frac{Q_1 Q_2}{R^2} \vec{a_R}$$

$$\therefore \vec{E} = k . \frac{Q_1 \cancel{Q_2}}{\cancel{Q_2} R^2} . \hat{a}_R = k . \frac{Q_1}{R^2} \hat{a}_R$$

But $k = \frac{1}{4\pi\varepsilon}$

$$\therefore \vec{E} = \frac{Q_1}{4\pi\varepsilon R^2} \vec{a_R} \quad V/m.$$

Unit of $\vec{E}$:

$\frac{\text{Newton}}{\text{Coulomb}}$ (N/C) (or) $\frac{\text{volts}}{\text{metre}}$ (V/m).

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Electric field ($\vec{E}$) due to various charge distributions :-

i) Point charge distribution, Q (C) :-

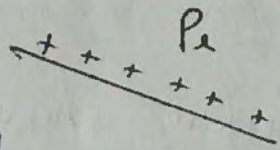
* volume of the point charge is zero because which do not occupy any space.

Electric field intensity due to point charge (Q_2) is given by

$$\vec{E}_2 = \frac{\vec{F}_{12}}{Q_2} = \frac{Q_1}{4\pi\varepsilon R_{12}^2} \vec{a}_{12}$$

ii) Line charge distribution, ρ_l (C/m) :-

charge is distributed along a line like a filament is called line charge distribution.



Line charge

Ex: Electron beam in CRT.

Line charge density (ρ_l)

ρ_l is defined as the total charge spread per unit length of the line

$$\rho_l = \frac{dQ}{dl} \quad C/m$$

$$\therefore dQ = \rho_l \, dl$$

$$Q = \int_l \rho_l \, dl$$

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Electric field due to Q at distance R is 6

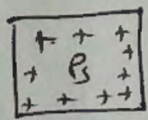
$$\vec{E} = \frac{Q}{4\pi\varepsilon R^2}\,\vec{a_R}$$

∴ Electric field due to line charge density is

$$\vec{E} = \frac{\int_l \rho_l \, dl}{4\pi\varepsilon R^2}\,\vec{a_R} \quad N/C \text{ or } V/m$$

iii) Surface charge distribution, ρ_s (C/m^2) :-

When a charge is confined to the surface of a conductor it is said to be surface charge distribution



surface charge

Ex :- conductor surface of a capacitor.

Surface charge density (ρ_s) :-

ρ_s is defined as the charge per unit area

$$\rho_s = \frac{dQ}{ds} \quad C/m^2$$

$$dQ = \rho_s \, ds$$

$$\therefore Q = \int_s \rho_s \, ds$$

$$\therefore \vec{E} = \frac{\int_s \rho_s \, ds}{4\pi\varepsilon R^2}\,\vec{a_R} \quad N/C$$

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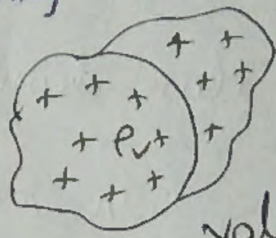
iv) volume charge distribution, ρ_v (c/m^3) ::

If the charge distributed uniformly in the volume is called volume charge.

Ex : Electron cloud in vaccum tube.



volume charge

Volume charge density (ρ_v) ::

ρ_v is defined as the charge per unit volume.

$$\rho_v = \frac{dQ}{dv} \quad c/m^3$$

$$dQ = \rho_v \, dv$$

$$\therefore Q = \int_v \rho_v \, dv$$

$$\therefore \vec{E} = \frac{\int_v \rho_v \, dv}{4\pi\varepsilon R^2} \vec{a_R}$$

* Electric flux density ::

The density of electric flux is called as electric flux density.

Electric flux density is defined as the ratio of electric flux to the surface area.

It is denoted by $\vec{D}$, vector quantity

$$\boxed{D = \frac{\varphi}{S}} = \frac{Q}{4\pi R^2} \quad ; \varphi - \text{Total flux}$$

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unit of $\vec{D}$: c/m^2

It is also called as Electric displacement flux density (or) displacement density.

In vector form

$$\vec{D} = \frac{d\psi}{ds} \vec{a_n} \quad c/m^2$$

<u>Relation between $\vec{E}$ and $\vec{D}$:-</u>

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$

$\varepsilon = \varepsilon_0 \varepsilon_r$; for free space $\varepsilon_r = 1$

$$\vec{D} = \varepsilon_0 \vec{E}$$

* <u>Electric flux density for various charge distributions :-</u>

i) <u>$\vec{D}$ due to point charge Q :</u>

Consider a point charge +Q placed at the centre of sphere of radius r.

$$\bar{D} = \frac{\psi}{S}$$

But
$\psi = Q$
$S = 4\pi r^2$

$$\therefore \vec{D} = \frac{Q}{4\pi r^2} \vec{a_r} \quad c/m^2$$

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ii) $\vec{D}$ due to line charge :

Consider a line charge having uniform charge density ρ_l C/m.

$$\therefore Q = \int_l \rho_l \, dl$$

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a_r} = \frac{\int_l \rho_l \, dl}{4\pi r^2} \vec{a_r}$$

If the line charge is infinite,

$$\vec{E} = \frac{\rho_l}{2\pi \varepsilon r} \vec{a_r} \quad V/m$$

$$\vec{D} = \frac{\rho_l}{2\pi r} \vec{a_r} \quad C/m^2$$

iii) $\vec{D}$ due to surface charge :

Consider a sheet of charge having uniform charge density ρ_s C/m^2.

$$Q = \int_s \rho_s \, ds$$

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a_r} = \frac{\int_s \rho_s \, ds}{4\pi r^2} \vec{a_r}$$

If the sheet charge is infinite,

$$\vec{E} = \frac{\rho_s}{2\varepsilon} \vec{a_r} \ V/m \ ; \quad \vec{D} = \frac{\rho_s}{2} \vec{a_r} \ C/m^2$$

iv) $\vec{D}$ due to volume charge :

Consider a charge enclosed by a volume with uniform charge density ρ_v C/m^3.

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$$Q = \int_v \rho_v \, dv$$

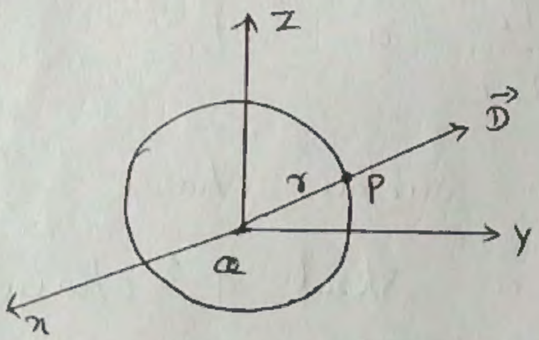
$$\vec{D} = \frac{\int_v \rho_v \, dv}{4\pi r^2} \vec{a_r}$$

* **Gauss's law :-**

"Gauss's law states that the net flux passing through any closed surface is equal to the charge enclosed by that surface".

$$\boxed{\psi = Q = \oint_s \vec{D} \cdot \vec{ds}}$$ => integral from of Gauss's law

Proof :-



consider a spherical surface with encloses charge Q.

The electric field intensity $\vec{E}$ at the spherical surface is

$$\vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \vec{a_r}$$

But $\vec{D} = \varepsilon_0 \vec{E} = \dfrac{Q}{4\pi r^2} \vec{a_r}$

Taking dot product with $\vec{ds}$ on both sides

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$$\vec{D}\cdot\vec{ds} = \frac{Q}{4\pi r^2}\vec{a_r}\cdot\vec{ds}$$

$$\vec{ds} = |\vec{ds}|\,\vec{a_r} = ds\,\vec{a_r}$$

$$\therefore \vec{D}\cdot\vec{ds} = \frac{Q}{4\pi r^2}\vec{a_r}\cdot ds\,\vec{a_r} \qquad \therefore \vec{a_r}\cdot\vec{a_r} = 1$$

$$\vec{D}\cdot\vec{ds} = \frac{Q}{4\pi r^2}ds$$

Taking surface integral on both sides

$$\oint_s \vec{D}\cdot\vec{ds} = \oint_s \frac{Q}{4\pi r^2}ds$$

$$= \frac{Q}{4\pi r^2}\oint_s ds$$

For sphere, $S = 4\pi r^2$

$$\therefore \oint_s \vec{D}\cdot\vec{ds} = \frac{Q}{\cancel{4\pi r^2}}\cdot\cancel{4\pi r^2}$$

Hence
$$\boxed{\oint_s \vec{D}\cdot\vec{ds} = Q}$$

Limitations of Gauss's law:

1. It cannot be applied on non-Gaussian surface
2. It can be applied only if the surface encloses the volume completely
3. It can applied only for symmetric charge distribution.

Application of Gauss's law:

i) It is used to find flux (ψ) and flux density ($\vec{D}$)

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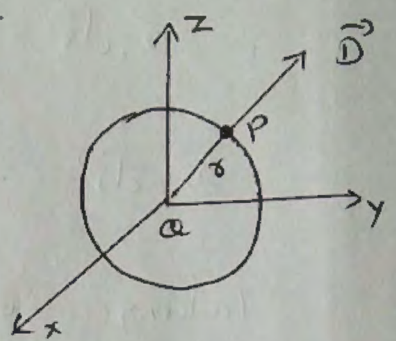
ii) It is used to find $\vec{E}$

iii) It is used to find the enclosed charge from the knowledge of either $\vec{E}$ or $\vec{D}$.

i) Field due to point charge:



Consider a point charge Q which is localed at the origin.

By applying Gauss's law to determine $\vec{D}$ at point P.

$$Q = \oint_S \vec{D}.\vec{ds}$$

But $\vec{ds} = ds\,\vec{a_r}$, $\vec{D} = D\,\vec{a_r}$

$$\therefore Q = \oint_S (D\,\vec{a_r}).(ds\,\vec{a_r}) \qquad \vec{a_r}.\vec{a_r} = 1$$

$$= D \oint_S ds$$

But

$$\oint_S ds = S = 4\pi r^2 = \text{surface area of sphere}$$

$$Q = D\,4\pi r^2$$

$$\therefore D = \frac{Q}{4\pi r^2}$$

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}}$$

and $\boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}}$ $\quad \therefore (\vec{D} = \varepsilon_0 \vec{E})$

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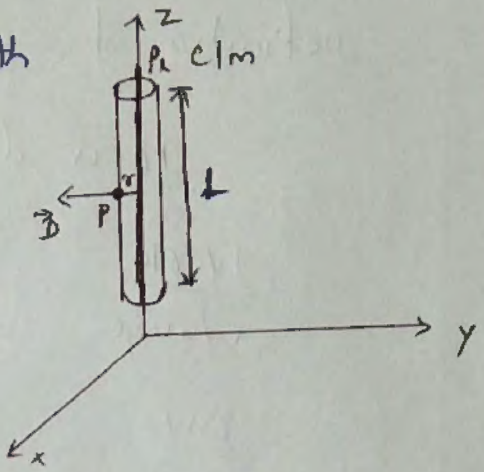
ii) Field due to infinite line charge :

Consider a conductor of length l carrying a uniform line charge density ρ_l C/m.



$$\therefore \rho_l = \frac{dQ}{dl} \quad C/m$$

$$dQ = \rho_l \cdot l \quad \text{Coulombs}$$

By applying Gauss's law

$$Q = \oint_s \vec{D} \cdot \vec{ds}$$

But

$$\vec{D} = D\,\vec{a_r} \quad ; \quad \vec{ds} = ds\,\vec{a_r}$$

$$\therefore Q = \oint_s D\,\vec{a_r} \cdot ds\,\vec{a_r} \qquad \vec{a_r} \cdot \vec{a_r} = 1$$

$$= D \oint_s ds$$

where $\oint_s ds = 2\pi r l \rightarrow$ Surface area of cylindrical Gaussian surface

$$\therefore Q = D\,(2\pi r l)$$

$$\rho_l \cancel{l} = D\,2\pi r \cancel{l}$$

$$\therefore D = \frac{\rho_l}{2\pi r}$$

$$\boxed{\vec{D} = \frac{\rho_l}{2\pi r}\vec{a_r}} \quad \text{and} \quad \boxed{\vec{E} = \frac{\rho_l}{2\pi \varepsilon_0 r}\vec{a_r}}$$

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* Electric Potential :-

Definition of Electric Scalar potential :-

It is defined as work per unit charge

$$\frac{\text{Work}}{\text{charge}} = \frac{\text{Force} \times \text{distance}}{\text{charge}}$$

$$\frac{\Delta W}{Q_0} = \frac{\Delta F_x \ \Delta x}{Q_0}$$

$$\therefore \ \Delta V = \frac{\Delta W}{Q_0} = E_x \Delta x$$

Unit of electric scalar potential :

$$\frac{\text{Joules}}{\text{coulomb}} = \frac{\text{volts}}{\cancel{\text{metre}}} \times \cancel{\text{metre}} = \text{volts.}$$

$$\frac{J}{C} = V.$$

* potential difference between two points :-

The potential of point A with respect to point B is V_{AB}.

$$V_{AB} = \frac{W}{Q}$$

But $W = -Q \int_B^A \vec{E} \cdot d\vec{l}$

$$\boxed{\therefore \ V_{AB} = - \int_B^A \vec{E} \cdot d\vec{l}}$$

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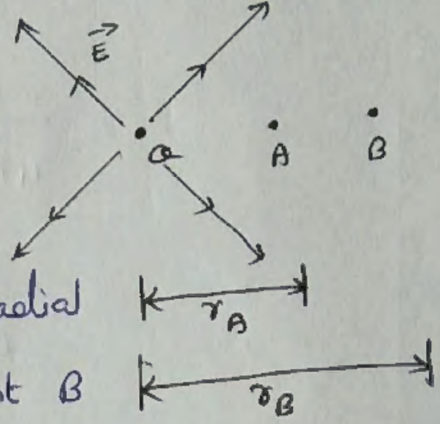
* Absolute electric potential (or) potential of a point charge :-



Consider a point charge Q is placed at origin of spherical coordinate system.

The point A is at the radial distance r_A from Q and the point B is at the radial distance r_B from Q.

The $\vec{E}$ of point charge Q is

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\vec{a_r} \rightarrow (1)$$

From fig. $r_B > r_A$

$$\therefore V_{AB} = -\int_{r_B}^{r_A} \vec{E}\cdot\vec{dl} \rightarrow (2)$$

But

$$\vec{dl} = dr\,\vec{a_r} + r\,d\theta\,\vec{a_\theta} + r\sin\theta\,d\phi\,\vec{a_\phi} \rightarrow (3)$$

Substitute eq's (3) and (1) in to eq (2),

$$V_{AB} = -\int_{r_B}^{r_A}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\,\vec{a_r}\right)\cdot\left(dr\,\vec{a_r} + r\,d\theta\,\vec{a_\theta} + r\sin\theta\,d\phi\,\vec{a_\phi}\right)$$

But $\vec{a_r}\cdot\vec{a_r} = 1$, $\vec{a_r}\cdot\vec{a_\theta} = \vec{a_r}\cdot\vec{a_\phi} = 0$

$$\therefore V_{AB} = -\int_{r_B}^{r_A}\frac{Q}{4\pi\varepsilon_0 r^2}\,dr$$

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$$V_{AB} = \frac{-Q}{4\pi\varepsilon_0}\int_{r_B}^{r_A}\frac{1}{r^2}\,dr$$

$$= \frac{-Q}{4\pi\varepsilon_0}\left(\frac{-1}{r}\right)_{r_B}^{r_A}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

$$\therefore V_{AB} = \frac{Q}{4\pi\varepsilon_0 r_A} - \frac{Q}{4\pi\varepsilon_0 r_B}$$

If $r_B > r_A$, V_{AB} is +ve. $(\because r_B = \infty)$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0 r_A} - \frac{1}{\infty} = \frac{Q}{4\pi\varepsilon_0 r_A}$$

If $V_{AB} = V_A - V_B$, $V_B = 0$

$$\therefore V_{AB} = V_A = \frac{Q}{4\pi\varepsilon_0 r_A} \text{ volts.}$$

This potential is called absolute potential at point r_A due to charge Q.

Calculation of potential differences for different configuration:

i) Potential due to infinite uniformly charged line:

consider an infinite line charge along z-axis having uniform charge density ρ_L C/m.

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The $\vec{E}$ due to infinite line is,

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r} \vec{a_r} \quad N/C. \rightarrow ①$$

The potential difference between point A & B is

$$V_{AB} = -\int_{r_B}^{r_A} \vec{E} \cdot \vec{dl} \rightarrow ②$$

But

$$\vec{dl} = dr\, \vec{a_r} \rightarrow ③$$

$$\therefore V_{AB} = -\int_{r_B}^{r_A} \frac{\rho_l}{2\pi\varepsilon_0 r} \vec{a_r} \cdot dr\, \vec{a_r} \qquad \because \vec{a_r} \cdot \vec{a_r} = 1$$

$$= -\frac{\rho_l}{2\pi\varepsilon_0} \int_{r_B}^{r_B} \frac{dr}{r}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0} (-\ln r)_{r_B}^{r_A}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0} (\ln r_B - \ln r_A)$$

$$\boxed{\therefore V_{AB} = \frac{\rho_l}{2\pi\varepsilon_0} \ln\left(\frac{r_B}{r_A}\right)} \text{ volts.} \rightarrow ③$$

iii) <u>Potential due to infinite sheet charge</u>:

Consider a infinite sheet charge along xy plane with uniform charge density ρ_s C/m^2.

$$\therefore \vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a_r}$$ ⇒ $\vec{E}$ due to infinite charged sheet

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$\therefore \quad V = -\int \vec{E} \cdot \vec{dl}$

$\vec{dl} = dr \, \vec{a_r}$

$V = -\int \frac{\rho_s}{2\varepsilon_0} \vec{a_r} \cdot dr \, \vec{a_r} \qquad \because \vec{a_r} \cdot \vec{a_r} = 1$

$= -\frac{\rho_s}{2\varepsilon_0} \int dr$

$$\boxed{V = -\frac{\rho}{2\varepsilon_0} r \text{ volts}}$$

iii) <u>Potential due to several charges</u> :-

Consider the charges $Q_1, Q_2 \text{----}, Q_n$ are placed at a distance $r_1, r_2 \text{----} r_n$ from point P respectively.

The total potential at point P is,

$V_p = V_1 + V_2 + \text{----} + V_n$

$= \frac{Q_1}{4\pi\varepsilon_0 r_1} + \frac{Q_2}{4\pi\varepsilon_0 r_2} + \text{----} + \frac{Q_n}{4\pi\varepsilon_0 r_n}$

$\therefore V_p = \frac{1}{4\pi\varepsilon_0} \left[\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \text{---} + \frac{Q_n}{r_n} \right]$

and also consider all charge distributions are present

$\therefore V_p = V_{point} + V_l + V_s + V_v$

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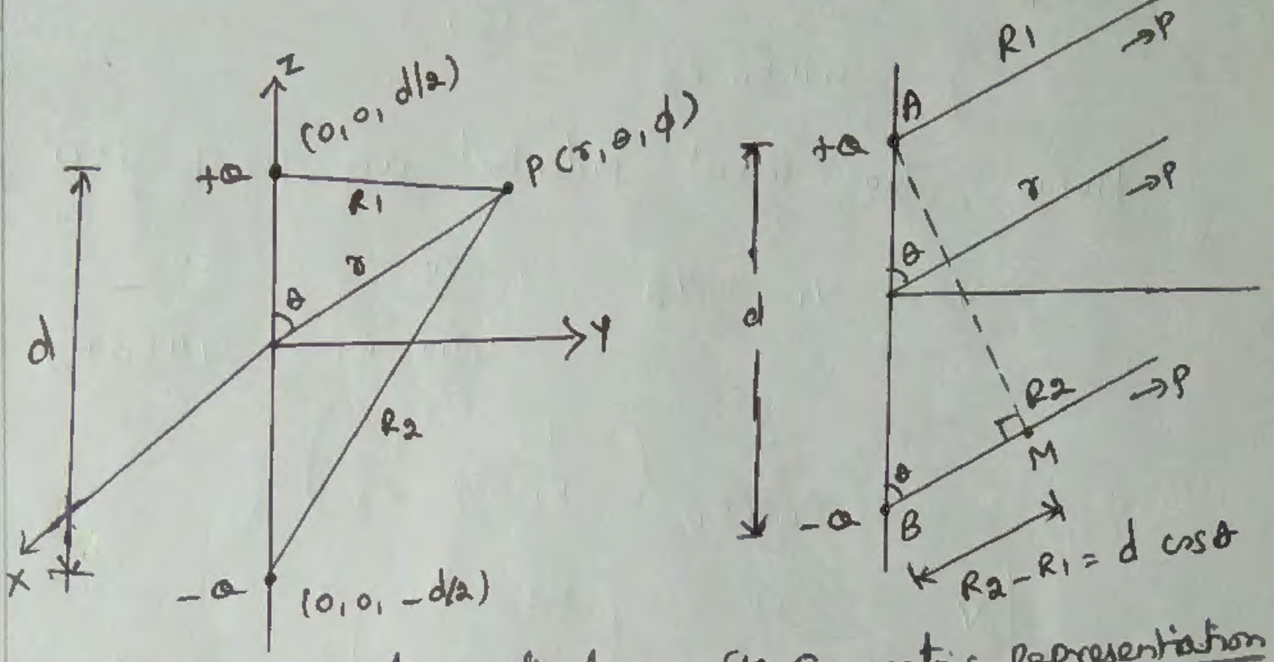
* Potential due to electric dipole :-



(a) Electric dipole (b) Geometric representation

Dipole : Two charges having equal in magnitude but opposite in sign is known as dipole.

Consider a electric dipole separated by a short distance d as shown in figure (a) and its geometry is shown in figure (b)

consider a spherical coordinate system at point P (r, θ, φ) at which we have to calculate electric field and electric potential.

According to definition of electric potential at point P due to +Q is

$$V_1 = \frac{+Q}{4\pi\varepsilon_0 R_1} \longrightarrow (1)$$

Potential due to -Q at point P is

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$$V_2 = -\frac{Q}{4\pi\varepsilon_0 R_2} \rightarrow (2)$$

Hence, the total potential at point P is

$$V = V_1 + V_2 = \frac{Q}{4\pi\varepsilon_0 R_1} - \frac{Q}{4\pi\varepsilon_0 R_2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right)} \rightarrow (3)$$

Now consider the following cases:

Case (i) :- consider point P is in xy plane (i.e., $z=0$) and $R_1 = R_2$

$\therefore \boxed{V = 0} \rightarrow$ Zero potential surface $\rightarrow (4)$

Case (ii) :- Consider the point P is far away from two point charges. (i.e, $R_1 \neq R_2$)

From Figure (b),

$\triangle AMB$, $\cos\theta = \frac{BM}{AB}$

$PB = PM + BM \Rightarrow BM = PB - PM$

$PA = PM = R_1 \qquad = R_2 - R_1$

$PB = R_2 \qquad \therefore AB \cos\theta = BM = R_2 - R_1$

$AB = d \qquad R_2 - R_1 = d\cos\theta \rightarrow (5)$

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Hence eq (5) becomes

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{R_1 R_2}\right)$$

If $R_1 \approx R_2 \approx r$

$$\therefore \boxed{V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{r^2}\right)} \rightarrow (6)$$

To find $\vec{E}$ at point P due to electric dipole:

$$\vec{E} = -\nabla V \rightarrow (7)$$

In spherical coordinate system,

$$\nabla V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi} \rightarrow (8)$$

From eq (8),

$$\vec{E} = -\left(\frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}\right) \rightarrow (9)$$

$$\frac{\partial V}{\partial r} = \frac{Q}{4\pi\varepsilon_0} d\cos\theta \frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)$$

$$= \frac{Qd\cos\theta}{4\pi\varepsilon_0}\left(-\frac{2}{r^3}\right) = -\frac{2Qd\cos\theta}{4\pi\varepsilon_0 r^3} \rightarrow (10)$$

$$\frac{\partial V}{\partial \theta} = \frac{Qd}{4\pi\varepsilon_0 r^2}\frac{\partial}{\partial \theta}(\cos\theta) = \frac{-Qd\sin\theta}{4\pi\varepsilon_0 r^2} \rightarrow (11)$$

and $\frac{\partial V}{\partial \phi} = 0 \rightarrow (12)$

Substitute eq's (10), (11) and (12) in to eq (9).

$$\vec{E} = \frac{2Qd\cos\theta}{4\pi\varepsilon_0 r^3}\vec{a_r} + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\vec{a_\theta}$$

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$$\therefore \boxed{\vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\vec{a_r} + \sin\theta\,\vec{a_\theta}\right)} \rightarrow (13)$$

Case (iii) :- Potential due to dipole moment :-

Dipole moment ($\bar{P}$) : It is defined as the product of charge and distance vector.

$$\therefore \bar{P} = Q\vec{d} \rightarrow (14)$$

Taking dot product of $\vec{a_r}$ on both sides

$$\vec{P}\cdot\vec{a_r} = Q\vec{d}\cdot\vec{a_r}$$

But $\vec{d}\cdot\vec{a_r} = |\vec{d}|\,|\vec{a_r}|\cos\theta$ $\quad \because |\vec{a_r}| = 1$

$$= d\cos\theta$$

$$\therefore \vec{P}\cdot\vec{a_r} = Qd\cos\theta$$

Hence eq (6) becomes

$$\boxed{V = \frac{\vec{P}\cdot\vec{a_r}}{4\pi\varepsilon_0 r^2}} \rightarrow (14)$$

---

* Relation between $\vec{E}$ and V :-

$$\vec{E} = -\nabla V$$

(or)

$$\nabla V = -\vec{E} \quad V/m$$

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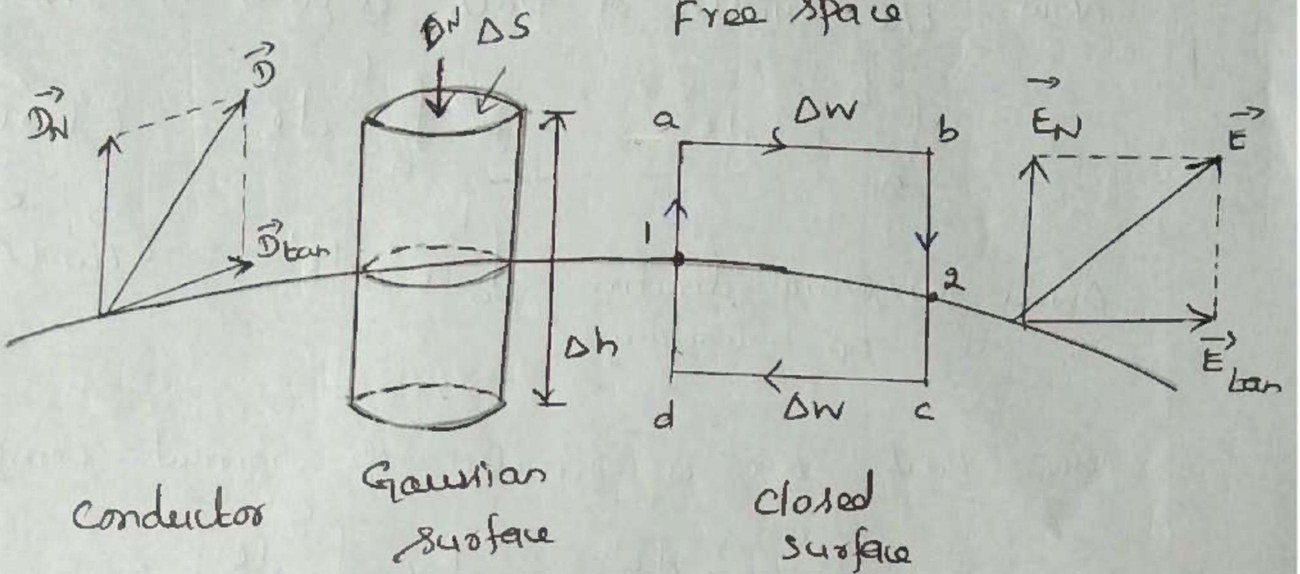
* **Boundary conditions for Electrostatic fields :-**

(i) **Boundary condition between conductor and free space :-**



consider boundary between conductor and free space. For ideal conductor, $\vec{E}$, $\vec{D}$ and ρ_v with in the conductor are zero.

To find tangential components of $\vec{E}$ and $\vec{D}$

For a closed surface (conservation field)

$$\oint \vec{E} \cdot d\vec{l} = 0 \quad \rightarrow (1)$$

Now consider rectangular closed path a-b-c-d-a as shown in Figure. $\oint \vec{E} \cdot d\vec{l}$ can be divided in to four parts

$$\oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l} = 0 \quad \rightarrow (2)$$

Now, the path c-d is in the conductor where $\vec{E} = 0$.

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Hence equation (2) becomes

$$\int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l} + \int_d^a \vec{E}\cdot d\vec{l} = 0 \rightarrow (3)$$

Now, the width of path a to b is Δw

$$\therefore \int_a^b \vec{E}\cdot d\vec{l} = \vec{E}\int_a^b d\vec{l} = \vec{E}(\Delta w)$$

Δw is tangential direction to boundary, $\int_a^b \vec{E}\cdot d\vec{l} = \vec{E}_{tan}(\Delta w)$ $\rightarrow (4)$

Now, path b c is parallel to normal component

$$\therefore \int_b^c \vec{E}\cdot d\vec{l} = E_N \int_b^c dl$$

But out of path b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E}=0$

$$\therefore \int_b^c \vec{E}\cdot d\vec{l} = E_N\left(\frac{\Delta h}{2}\right) \rightarrow (5)$$

Similarly, path d-a (same as path b-c, only direction is opposite)

$$\int_d^a \vec{E}\cdot d\vec{l} = -E_N \frac{\Delta h}{2} \rightarrow (6)$$

substitute eq's (4), (5) & (6) in (3)

$$\vec{E}_{tan}\,\Delta w + E_N \frac{\Delta h}{2} - E_N \frac{\Delta h}{2} = 0$$

$$\therefore \boxed{\vec{E}_{tan} = 0} \rightarrow (7)$$

But $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \boxed{\vec{D}_{tan} = 0} \rightarrow (7)$$

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To find normal components of $\vec{E}$ and $\vec{D}$:-

Consider a Gaussian surface to find normal components of $\vec{E}$ and $\vec{D}$.

According to Gauss's law

$$\oint_s \vec{D} \cdot \vec{ds} = Q \rightarrow (9)$$

The surface integral must be evaluated over three surfaces

$$\int_{Top} \vec{D} \cdot \vec{ds} + \int_{Bottom} \vec{D} \cdot \vec{ds} + \int_{Lateral} \vec{D} \cdot \vec{ds} = Q \rightarrow (10)$$

Now, bottom surface is in the conductor where $\vec{E} = 0$

$$\therefore \oint_{Top} \vec{D} \cdot \vec{ds} + \int_{Lateral} \vec{D} \cdot \vec{ds} = Q \rightarrow (11)$$

But area of the surface is very small, $\Delta S \rightarrow 0$

$$\therefore \int_{Lateral} \vec{D} \cdot \vec{ds} = 0$$

Hence eq (11) becomes

$$\int_{Top} \vec{D} \cdot \vec{ds} = Q$$

$$D_N \, \Delta S = Q \qquad \therefore (Q = \rho_s \Delta S)$$

$$\therefore D_N \Delta S = \rho_s \Delta S$$

$$\therefore \boxed{D_N = \rho_s} \rightarrow (12) \qquad \therefore D_N = \varepsilon_0 E_N$$

$$\boxed{E_N = \rho_s / \varepsilon_0} \rightarrow (13)$$

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ii) Boundary condition between conductor and dielectric :-

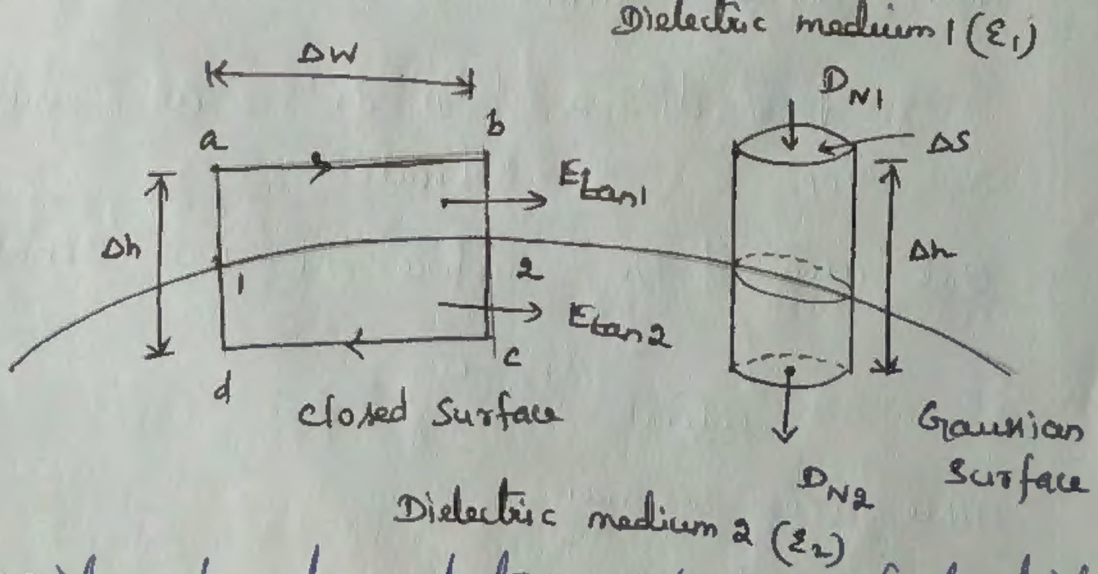
For dielectric ε,

$$\varepsilon = \varepsilon_0 \varepsilon_r$$

Replace ε_0 by ε_0 in eq (13)

$$\therefore \boxed{\begin{aligned} E_{tan} &= D_{tan} = 0 \\ D_N &= \rho_s \\ E_N &= \frac{\rho_s}{\varepsilon} = \frac{\rho_s}{\varepsilon_0 \varepsilon_r} \end{aligned}}$$

iii) Boundary condition for perfect dielectric material :-



Consider boundary between two perfect dielectrics having permittivities ε_1 and ε_2.

To find tangential components :-

For a closed surface (conservative field)

$$\oint_l \vec{E} \cdot \vec{dl} = 0 \quad \rightarrow (1)$$

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consider a closed path a-b-c-d-a,

$$\int_a^b \vec{E}\cdot d\vec{l} + \int_b^c \vec{E}\cdot d\vec{l} + \int_c^d \vec{E}\cdot d\vec{l} + \int_d^a \vec{E}\cdot d\vec{l} = 0 \quad \rightarrow (2)$$

For paths b-c & d-a, the height $\Delta h \rightarrow 0$

Hence eq (2) becomes

$$\int_a^b \vec{E}\cdot d\vec{l} + \int_c^d \vec{E}\cdot d\vec{l} = 0 \quad \rightarrow (3)$$

Now path a-b is in medium 1, hence corresponding component of $\vec{E}$ is E_{tan1}

$$\therefore \int_a^b \vec{E}\cdot d\vec{l} = E_{tan1}\int_a^b d\vec{l} = E_{tan1}\Delta w \quad \rightarrow (4)$$

similarly, for path c-d, (medium 2)

$$\int_c^d \vec{E}\cdot d\vec{l} = -E_{tan2}\int_c^d d\vec{l} = -E_{tan2}\Delta w \quad \rightarrow (5)$$

Substitute eq's (4) and (5) into (3)

$$\therefore E_{tan1}\Delta w - E_{tan2}\Delta w = 0$$

$$E_{tan1} - E_{tan2} = 0$$

$$\therefore \boxed{E_{tan1} = E_{tan2}} \quad \rightarrow (6)$$

But $\vec{D} = \varepsilon \vec{E}$

$$\therefore D_{tan1} = \varepsilon_1 E_{tan1} \text{ and } D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

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$$\therefore \boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}} \rightarrow (7)$$

To find normal components of $\bar{E}$ and $\bar{D}$:-

Not According to Gauss's law

$$\oint \vec{D} \cdot \vec{ds} = Q \rightarrow (8)$$

Consider a Gaussian surface

$$\int_{Top} \vec{D} \cdot \vec{ds} + \int_{Bottom} \vec{D} \cdot \vec{ds} + \int_{Lateral} \vec{D} \cdot \vec{ds} = Q \rightarrow (9)$$

For Lateral surface, $\Delta h \rightarrow 0$

$$\therefore \int_{Top} \vec{D} \cdot \vec{ds} + \int_{Bottom} \vec{D} \cdot \vec{ds} = Q \rightarrow (10)$$

For top surface, the direction of D_{N1} is entering the boundary while bottom surface, the direction of D_{N2} is leaving the boundary.

$$\therefore \int_{Top} \vec{D} \cdot \vec{ds} = D_{N1} \int_{Top} \vec{ds} = D_{N1} \Delta S \rightarrow (11)$$

$$\int_{Bottom} \vec{D} \cdot \vec{ds} = D_{N2} \int_{Bottom} \vec{ds} = -D_{N2} \Delta S \rightarrow (12)$$

Hence, eq (10) can be reduced to

$$D_{N1} \Delta S - D_{N2} \Delta S = Q$$

$$(D_{N1} - D_{N2}) \Delta S = Q$$

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But $Q = \rho_s \Delta S$

$$\therefore (D_{N1} - D_{N2}) \cancel{\Delta S} = \rho_s \cancel{\Delta S}$$

$$D_{N1} - D_{N2} = \rho_s$$

For ideal dielectric media, $\rho_s = 0$

$$\therefore \boxed{D_{N1} = D_{N2}} \rightarrow (13)$$

But $D_{N1} = \varepsilon_1 E_{N1}$ and $D_{N2} = \varepsilon_2 E_{N2}$

$$\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$$

$$\therefore \boxed{\frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1}} \rightarrow (14)$$

---

* <u>Capacitance</u> :-

Capacitance is defined as the ratio of the magnitude of the total charge to the potential difference between the conductors.

$$\therefore C = \frac{Q}{V} \text{ Farads (F)}$$

$$= \frac{\oint_S \varepsilon \vec{E} \cdot \vec{ds}}{-\int_-^+ \vec{E} \cdot \vec{dl}} \text{ Farads.}$$

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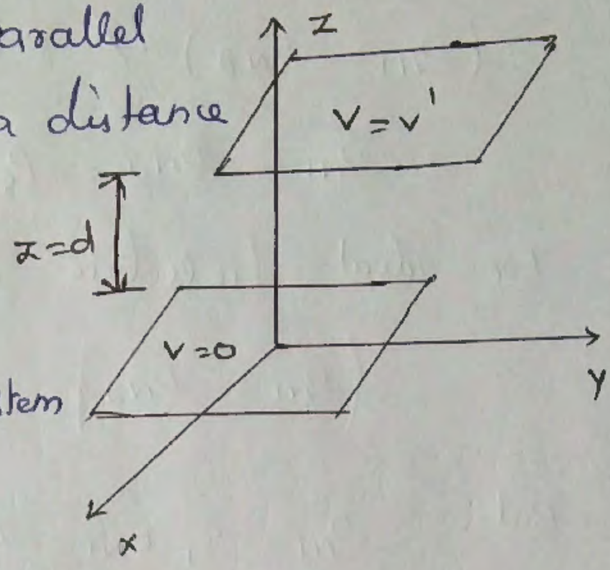
* <u>Capacitance of parallel plate capacitor</u> :-

It consist of two parallel plates, separated by a distance 'd'.



Laplace equation in cartesian coordinate system

$$\nabla^2 V = 0$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \rightarrow (1)$$

From Fig, voltage is a function z only.

$$\therefore \frac{\partial^2 V}{\partial z^2} = 0 \rightarrow (2)$$

Integrating eq (2),

$$\int \frac{\partial^2 V}{\partial z^2} = \frac{\partial V}{\partial z} = C_1 \rightarrow (3)$$

Integrating eq (3),

$$\int \frac{\partial V}{\partial z} = V = C_1 z + C_2 \rightarrow (4)$$

By applying boundary conditions,

$z = 0$ at $V = 0$ and $z = d$ at $V = V'$

$$C_2 = 0 \rightarrow (5)$$

$$V' = C_1 d + 0$$

$$C_1 = V'/d \rightarrow (6)$$

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Substitute eq's (5) & (6) in to (4),

$$V = \frac{V'}{d} z \rightarrow (7)$$

Eq (7) gives the potential at any point between the plates.

Now, the relation between $\vec{E}$ & V is

$$\vec{E} = -\nabla V$$

$$\vec{E} = -\left(\frac{\partial v}{\partial x}\vec{a_x} + \frac{\partial v}{\partial y}\vec{a_y} + \frac{\partial v}{\partial z}\vec{a_z}\right) \rightarrow (8)$$

But voltage is varying along z-axis only

$$\therefore \vec{E} = -\frac{\partial v}{\partial z}\vec{a_z} \rightarrow (9)$$

From eq's (3) and (6),

$$\frac{\partial v}{\partial z} = c_1 \quad ; \quad c_1 = V'/d$$

$$\therefore \frac{\partial v}{\partial z} = \frac{V'}{d}$$

$$\therefore \vec{E} = -\frac{V'}{d}\vec{a_z}$$

But $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \vec{D} = -\varepsilon_0 \frac{V'}{d}\vec{a_z} \rightarrow (10)$$

For parallel plate capacitor, the magnitude of flux density is equal to surface charge density on the plate of capacitor.

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$$\therefore |\vec{D}| = \rho_s = \varepsilon_0 \frac{V'}{d} \rightarrow (11)$$

According to Gauss's law

$$Q = \oint_s \rho_s \, ds = \oint_0 \varepsilon_0 \frac{V'}{d} d$$

$$= \varepsilon_0 \frac{V'}{d} \oint_s ds = \varepsilon_0 \frac{V'}{d} A \rightarrow (12)$$

where

$\oint ds$ = Surface area of plate = A

From eq's (7) and (12),

$$\therefore C = \frac{|Q|}{V}$$

$$= \frac{\varepsilon_0 \cancel{V'}}{\cancel{d}} \cdot A \cdot \frac{\cancel{d}}{\cancel{V'} z}$$

$$C = \frac{\varepsilon_0 A}{z}$$

From Fig, $\therefore z = d$

$$\therefore \boxed{C = \frac{\varepsilon_0 A}{d}}$$

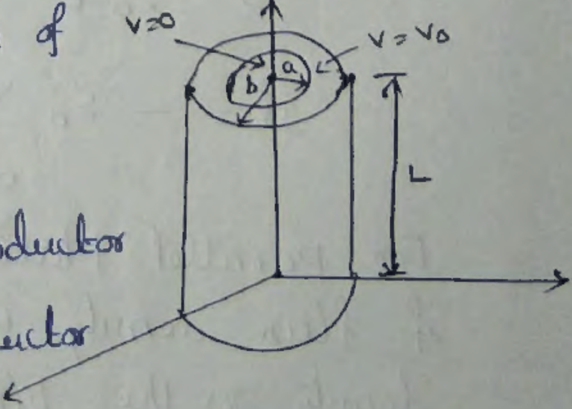
In general

$$C = \frac{\varepsilon A}{d} = \frac{\varepsilon_0 \varepsilon_r A}{d} \text{ Farads.} \quad \because \varepsilon = \varepsilon_0 \varepsilon_r$$

---

* Capacitance of coaxial cable capacitor:

A coaxial cable consists of two cylindrical conductors.



a = Radius of inner conductor

b = Radius of outer conductor

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In cylindrical coordinates, the Laplace equation is

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \rightarrow (1)$$

Now in case of coaxial cable, the voltage is in the radial direction

$$\therefore \frac{\partial^2 V}{\partial\phi^2} = \frac{\partial^2 V}{\partial z^2} = 0$$

Eq (1) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad \rightarrow (2)$$

Integrating eq (2),

$$\rho\frac{\partial V}{\partial\rho} = C_1$$

$$\frac{\partial V}{\partial\rho} = \frac{C_1}{\rho} \quad \rightarrow (3)$$

Integrating eq (3),

$$V = \int \frac{C_1}{\rho}\,d\rho$$

$$V = C_1 \ln\rho + C_2 \quad \rightarrow (4)$$

Applying boundary conditions

$V = 0$ at $\rho = a$

$V = V_0$ at $\rho = b$

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Hence eq ④ becomes

$$0 = c_1 \ln a + c_2$$

$$c_2 = -c_1 \ln a \rightarrow ⑤$$

and

$$V_0 = c_1 \ln b + c_2 \rightarrow ⑥$$

Substitute eq ⑤ into ⑥,

$$V_0 = c_1 \ln b - c_1 \ln a$$

$$= c_1 \ln (b/a)$$

$$\therefore c_1 = \frac{V_0}{\ln(b/a)} \rightarrow ⑦$$

Substitute eq ⑦ into ⑤,

$$c_2 = \frac{-V_0}{\ln(b/a)} \cdot \ln a \rightarrow ⑧$$

Substitute eq ⑦ and ⑧ into ④,

$$V = \frac{V_0}{\ln(b/a)} \ln \rho - \frac{V_0}{\ln(b/a)} \ln a$$

$$= \frac{V_0}{\ln(b/a)} [\ln \rho - \ln a]$$

$$\boxed{V = V_0 \frac{\ln(\rho/a)}{\ln(b/a)}} \rightarrow ⑨$$

Eq ⑨ gives the equation of voltage between two cylinders

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Now in cylindrical coordinate system, $\vec{E}$ and V are related as

$$\vec{E} = -\left(\frac{\partial v}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial v}{\partial \phi}\vec{a_\phi} + \frac{\partial v}{\partial z}\vec{a_z}\right)$$

$$\vec{E} = -\frac{\partial v}{\partial \rho}\vec{a_\rho} \quad \rightarrow (10)$$

From eq (3), and (7)

$$\vec{E} = -\frac{c_1}{\rho} = -\frac{V_0}{\rho}\frac{1}{\ln(b/a)}\vec{a_\rho}$$

$$\therefore \vec{D} = -\varepsilon_0 V_0/\rho \times \ln(b/a)\ \hat{a}_\rho$$

But $|\vec{D}| = \rho_s$ = Surface charge density

$$= \frac{\varepsilon_0 V_0}{\rho}\cdot\frac{1}{\ln(b/a)} \quad \rightarrow (11)$$

Using Gauss's law

$$Q = \oint_s \rho_s\, ds = \rho_s \oint ds = \rho_s \cdot 2\pi\rho L \quad \rightarrow (12)$$

Where

$\oint_s ds$ = Area of cylinder = $\rho_s \cdot 2\pi r L$ $\quad (r = \rho)$

$$\therefore Q = \frac{\varepsilon_0 V_0}{\rho}\frac{1}{\ln(b/a)}\cdot 2\pi\rho L$$

$$= \frac{2\pi\varepsilon_0 V_0 L}{\ln(b/a)} \quad \rightarrow (13)$$

$$\therefore C = \frac{|Q|}{V} = \frac{2\pi\varepsilon_0 \cancel{V_0} L}{\ln(b/a)}\cdot\frac{1}{\cancel{V_0}} \qquad V = V_0$$

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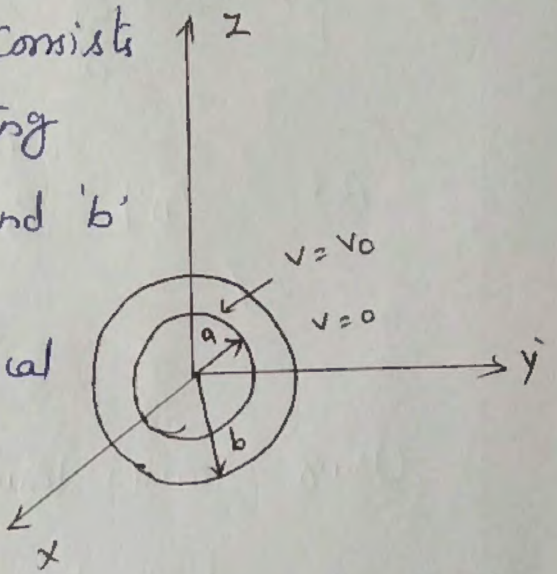
$$\boxed{C = \frac{2\pi \varepsilon_0 L}{\ln (b/a)}} \text{ Farads} \rightarrow (14)$$

In general, $\varepsilon = \varepsilon_0 \varepsilon_r$

$$C = \frac{2\pi \varepsilon_0 \varepsilon_r L}{\ln(b/a)} = \frac{2\pi \varepsilon L}{\ln(b/a)}$$

* Capacitance of Spherical Capacitor :-

A Spherical capacitor Consists of two concentric conducting spheres with radii 'a' and 'b'



Laplace's equation in Spherical coordinate system as a function of radius only.

$$\nabla^2 v = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) = 0$$

$$\therefore \frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) = 0 \rightarrow (1)$$

By integrating eq (1)

$$r^2 \frac{\partial v}{\partial r} = C_1$$

$$\frac{\partial v}{\partial r} = \frac{C_1}{r^2}$$

$$\partial v = \frac{C_1}{r^2} \partial r \rightarrow (2)$$

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By integrating eq ②,

$$\int \partial v = \int \frac{c_1}{r^2} \partial r$$

$$v = c_1\left(\frac{-1}{r}\right) + c_2$$

$$v = -\frac{c_1}{r} + c_2 \rightarrow ③$$

By applying boundary conditions,

$v = V_0$ at $r = a$

$V = 0$ at $r = b$

$$0 = -\frac{c_1}{b} + c_2 \rightarrow ④$$

$$V_0 = -\frac{c_1}{a} + c_2 \rightarrow ⑤$$

Subtracting eq ④ from ⑤

$$V_0 = -\frac{c_1}{a} + \frac{c_1}{b} = c_1\left[\frac{1}{b} - \frac{1}{a}\right]$$

$$V_0 = c_1\left(\frac{a-b}{ab}\right)$$

$$c_1 = V_0\left(\frac{ab}{a-b}\right) \rightarrow ⑥$$

Substitute eq ⑥ into ④

$$0 = -\frac{V_0}{b}\left(\frac{ab}{a-b}\right) + c_2$$

$$c_2 = \frac{V_0}{b}\left(\frac{ab}{a-b}\right) = V_0\left(\frac{a}{a-b}\right) \rightarrow ⑦$$

Substitute eq's ⑥ and ⑦ into ③,

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$$V = -\frac{V_0}{r}\left(\frac{ab}{a-b}\right) + V_0\left(\frac{a}{a-b}\right) \rightarrow (8)$$

Eq (8) gives the potential between the spheres.

Now, in spherical coordinate system $\vec{E}$ and V are related as

$$\vec{E} = -\frac{\partial V}{\partial r}\vec{a_r}$$

But $\frac{\partial V}{\partial r} = \frac{C_1}{r^2}$

From eq (6)

$$\therefore \vec{E} = -\frac{V_0}{r^2}\left(\frac{ab}{a-b}\right)\vec{a_r}$$

But $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \vec{D} = -\varepsilon_0 \frac{V_0}{r^2}\left(\frac{ab}{a-b}\right)\vec{a_r}$$

But $|\vec{D}| = \rho_s = \frac{\varepsilon_0 V_0}{r^2}\left(\frac{ab}{a-b}\right)$

Using Gauss's law

$$Q = \oint_S \rho_s \, ds = \rho_s \oint_S ds$$

$$= \rho_s \times \text{Area of Sphere}$$

$$= \rho_s \times 4\pi r^2$$

$$\therefore Q = \frac{\varepsilon_0 V_0}{\cancel{r^2}}\left(\frac{ab}{a-b}\right) \times 4\pi \cancel{r^2}$$

$$= 4\pi \varepsilon_0 V_0 \left(\frac{ab}{a-b}\right)$$

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$$\therefore C = \frac{Q}{V} = \frac{Q}{V_0} \qquad \because V = V_0$$

$$C = 4\pi \varepsilon_0 \cancel{V_0} \frac{ab}{(a-b)} \cdot \frac{1}{\cancel{V_0}}$$

$$\therefore \boxed{C = 4\pi \varepsilon_0 \left(\frac{ab}{a-b}\right)}$$

in general

$$C = 4\pi \varepsilon_0 \varepsilon_r \left(\frac{ab}{a-b}\right)$$

---

* Poisson's and Laplace's Equation :-

Laplace's equation gives the method of finding potential function V.

Poisson's equation is obtained from point form of Gauss's law

$$\nabla \cdot \vec{D} = \rho_v \rightarrow (1)$$

w.k.t, $\vec{D} = \varepsilon \vec{E}$

$$\therefore \nabla \cdot (\varepsilon \vec{E}) = \rho_v \rightarrow (2)$$

we have relation

$$\vec{E} = -\nabla V \rightarrow (3)$$

Eq (2) becomes

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v$$

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$$-\nabla\cdot(\varepsilon\nabla V) = \rho_v \rightarrow (4)$$

If the medium is homogeneous, ε is constant

$$\therefore \nabla\cdot\nabla V = -\frac{\rho_v}{\varepsilon}$$

But $\nabla\cdot\nabla = \nabla^2$

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \rightarrow (5)$$

Eq (5) is called as Poisson's equation.

Now, if $\rho_v = 0$ but there is a presence of point, line charge or surface charge, eq (5) becomes

$$\boxed{\nabla^2 V = 0} \rightarrow (6)$$

Eq (6) is called as Laplace's equation

In cartesian coordinate system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In cylindrical coordinate system

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In spherical coordinate system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial z^2} = 0$$

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Application of Poisson's and Laplace's equation: (41)

1. Laplace and poisson equations are used to solving electrostatic problems
   Ex: capacitors and vacuum tube diodes
2. Also used in other field problems
   Ex: Magnetostatics, temperature in heat conduction, pressure head in seepage.

---

* **conductors in static electric field:**

A conductor consists of large number of free electrons which constitute conduction current with the application of an electric field.

No forbidden energy gap between valance and conduction band.

In perfect conductor,

$$\sigma = \infty \quad \text{or} \quad \frac{\sigma}{\omega \varepsilon} >> 1$$

properties of conductors, under static conditions are

i, Electric field ($\vec{E} = 0$) is zero with in a conductor

ii, Charge density is zero ($\rho_v = 0$) within a conductor

iii, The Conductivity is infinite ($\sigma = \infty$).

Some conductors exhibit infinite conductivity are called super conductors. Ex: Lead & aluminium.

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* Dielectrics in Static electric field :-

Ideal dielectric does not contain free electrons, in which charges are well bounded and cannot be set in motion easily.

Large forbidden energy gap between valance band and conduction band.

In perfect dielectrics,

$$\sigma = 0 \quad \text{or} \quad \frac{\sigma}{\omega \varepsilon} << 1$$

It does not conduct current & opposes the flow of current

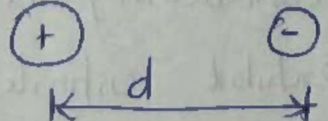
Properties of dielectric materials are:

1. Electric & magnetic fields exists in a dielectric material
2. Volume charge density is zero $(\rho_v = 0)$
3. conductivity is zero $(\sigma = 0)$.

Dielectrics are classified in to 2 types

i) Polar type materials : centres of +ve and -ve charge molecules are separated by a small distance. It acts as a electric dipole

Ex: water, hydrochloric acid



Polar type

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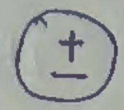
ii) Non-polar type: Centres of +ve and -ve charges of molecule are coincide.

Ex: Oxygen and hydrogen


Non Polar Type

* **current** :-

It is defined as charge passing through the medium per unit time.

$$I = \frac{dQ}{dt} \text{ Amperes}$$

The current is a scalar quantity.

Three types of currents are:

i) **Convection current:**

It is defined as the current produced by a beam of electrons flowing through an insulating medium. Ex: Current through a vacuum tube

ii) **Conduction current:-**

current produced due to flow of electrons in a conductor. This obeys ohm's law

iii) **Displacement current:**

current flows as a result of time varying electric field in a dielectric material

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* current density : ($\vec{J}$)

current density is defined as current through a unit normal area.

$$\vec{J} = \frac{dI}{ds} \quad \text{or} \quad \vec{J} = \frac{I}{S}\vec{a_n} \quad A/m^2$$

* convection current density :

It is related to volume charge density and velocity vector

$$\therefore \vec{J} = \rho_v \vec{v} \quad A/m^2$$

* Conduction current density :

Conduction current density exists in the case of conductors when an electric field applied.

$$\vec{J_c} = \sigma \vec{E} \quad A/m^2 \qquad \sigma = -\rho_e \mu_e$$

* Displacement current density :

It is defined as the rate of displacement electric flux density with time.

$$\vec{J_d} = \frac{\partial \vec{D}}{\partial t} \quad A/m^2$$

* Resistance of a conductor :

The ratio of potential difference between the two ends of conductors to the current

$$R = \frac{V}{I} = \frac{L}{\sigma S} = \frac{\rho_c L}{S}\ \Omega \ ; \ \rho_c = \frac{1}{\sigma}\ \Omega\text{-}m$$

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* Continuity Equation:

The differential equation relating the current density $\vec{J}$ and volume charge density ρ_v at each point in a closed surface area is known as continuity equation.

This equation based on the law of conservation of charge in closed surface.

The current through the closed surface is

$$I = \oint_S \vec{J} \cdot \vec{ds} \rightarrow (1)$$

But $I = -\frac{dQ}{dt}$, eq (1) becomes

$$\therefore \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt} \rightarrow (2)$$

=> Integral form of continuity equation

Using Gauss's law

$$Q = \int_V \rho_v \, dv$$

$$\therefore \oint_S \vec{J} \cdot \vec{ds} = -\int_V \frac{\partial \rho_v}{\partial t} \, dv$$

By applying divergence theorem

$$\oint_S \vec{J} \cdot \vec{ds} = \int_V (\nabla \cdot \vec{J}) \, dv$$

$$\therefore \int_V (\nabla \cdot \vec{J}) \, dv = -\int_V \frac{\partial \rho_v}{\partial t} \, dv$$

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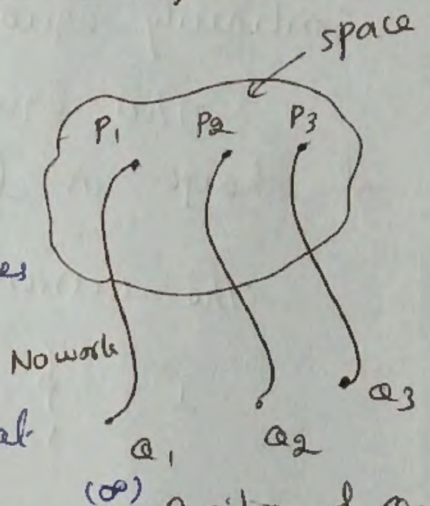
$$\therefore \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \rightarrow (3)$$

Eq (3) is called Point form (or) differential form of continuity equation.

* **Electrostatic energy and energy density:-**



Position of Q_1, Q_2 and Q_3 in space

Consider empty space, Q_1 is moved from infinity to a point in space say P_1, this requires no work.

If the charge Q_2 is placed at P_2, work is required to move against the field of Q_1.

$\therefore$ Workdone = potential × charge

Workdone to position Q_2 at P_2 = $V_{21}\, Q_2 \rightarrow (1)$

where V_{21} = Potential at P_2 due to P_1.

Workdone to position Q_3 at P_3 = $V_{31} Q_3 + V_{32} Q_3 \rightarrow (2)$

$\therefore$ Workdone to position Q_n at P_n = $V_{n1}\, Q_n + V_{n2}\, Q_n + V_{n3}\, Q_n + \cdots \rightarrow (3)$

$\therefore$ Total workdone is given by

$$W_E = Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + \cdots \rightarrow (4)$$

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If the charges are placed in reverse order, (47)

$$W_E = Q_3 V_{34} + Q_2 V_{23} + Q_2 V_{24} + \cdots Q_1 V_{1n} \rightarrow (5)$$

Adding eq's (4) and (5),

$$2W_E = Q_1 (V_{12} + V_{13} + V_{14} + \cdots + V_{1n}) +$$
$$Q_2 (V_{21} + V_{23} + V_{24} + \cdots + V_{2n}) +$$
$$Q_3 (V_{31} + V_{32} + V_{34} + \cdots + V_{3n}) \rightarrow (6)$$

where

$$V_1 = V_{12} + V_{13} + V_{14} + \cdots V_{1n}$$

Hence eq (6) becomes

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \cdots$$
$$\cdots + Q_n V_n \rightarrow (7)$$

$$\therefore W_E = \frac{1}{2} \sum_{m=1}^{n} Q_m V_m \rightarrow (8)$$

, Joules

For continuous charge distributions,

$$W_E = \frac{1}{2} \int_v \rho_v \, v \, dv . \rightarrow (9)$$

According to point form of Gauss's law

$$\nabla \cdot \vec{D} = \rho_v \rightarrow (10)$$

Eq (10) becomes

$$W_E = \frac{1}{2} \int_v (\nabla \cdot \vec{D}) \, v \, dv \rightarrow (11)$$

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By using vector identity,

$$\nabla \cdot V\vec{A} = \vec{A}\,\nabla V + V(\nabla \cdot \vec{A})$$

$$\therefore (\nabla \cdot \vec{A})V = \nabla \cdot V\vec{A} - \vec{A} \cdot \nabla V \rightarrow (12)$$

Using eq (12) in eq (11),

$$W_E = \frac{1}{2}\int_V (\nabla \cdot V\vec{D} - \vec{D} \cdot \nabla V)\, dv$$

$$= \frac{1}{2}\int_V \nabla \cdot V\vec{D}\, dv - \frac{1}{2}\int_V \vec{D} \cdot \nabla V\, dv \rightarrow (13)$$

According to divergence theorem

$$\frac{1}{2}\int_V \nabla \cdot V\vec{D}\, dv = \frac{1}{2}\oint_S (V\vec{D}) \cdot d\vec{s}$$

$$\therefore W_E = \frac{1}{2}\oint_S (V\vec{D}) \cdot d\vec{s} - \frac{1}{2}\int_V (\vec{D} \cdot \nabla V)\, dv \rightarrow (14)$$

If surface becomes very large, closed surface integral is zero

$$\therefore W_E = -\frac{1}{2}\int_V (\vec{D} \cdot \nabla V)\, dv$$

But $\vec{E} = -\nabla V$

$$W_E = -\frac{1}{2}\int_V \vec{D} \cdot (-\vec{E})\, dv$$

$$\boxed{W_E = \frac{1}{2}\int_V \vec{D} \cdot \vec{E}\, dv} \quad \text{Joules}$$

But $\vec{D} = \varepsilon_0 \vec{E}$

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$$\therefore \quad W_E = \frac{1}{2}\int_v \varepsilon_0 E^2 \, dv$$

$$W_E = \frac{1}{2}\int_v \frac{D^2}{\varepsilon_0} \, dv$$

In differential form,

$$dW_E = \frac{1}{2} \vec{D} \cdot \vec{E} \, dv$$

$$\frac{dW_E}{dv} = \frac{1}{2} \vec{D} \cdot \vec{E} \quad J/m^3 \rightarrow \text{Energy density in electric field}$$

---

* Uniqueness of electrostatic solutions :-

(or)

Uniqueness theorem :-

Statement :- "If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique".

Laplace's equation, $\nabla^2 V = 0$

Assume that there are two solutions V_1 and V_2

$$\left.\begin{array}{l} \therefore \nabla^2 V_1 = 0 \quad \& \quad \nabla^2 V_2 = 0 \\ V_1 - V_2 \text{ on the boundary} \end{array}\right\} \rightarrow (1)$$

Now we consider the difference

$$V_d = V_2 - V_1 \rightarrow (2)$$

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$$\therefore \nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \rightarrow (3)$$

$$\therefore V_d = 0 \text{ on the boundary} \rightarrow (4)$$

From the divergence theorem,

$$\int_v \nabla \cdot A \, dv = \oint_s A \cdot ds \rightarrow (5)$$

$$\text{Let } A = V_d \nabla V_d \rightarrow (6)$$

$$\nabla \cdot A = \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d \rightarrow (7)$$

From eq (3), $\nabla^2 V_d = 0$

$$\therefore \nabla \cdot A = \nabla V_d \cdot \nabla V_d \rightarrow (7)$$

Substitute eq (7) in (5)

$$\int_v \nabla V_d \cdot \nabla V_d \, dv = \oint_s V_d \nabla V_d \cdot ds \rightarrow (8)$$

From eq (4) and (7), RHS of eq (8) becomes zero

$$\therefore \int_v \nabla V_d \cdot \nabla V_d \, dv = 0$$

$$\int_v |\nabla V_d|^2 \, dv = 0$$

$$\therefore \nabla V_d = 0 \text{ or } V_d = \text{constant}$$
$$= V_1 - V_2$$

This shows that both the assumed solutions are one and the same and hence solution is unique.

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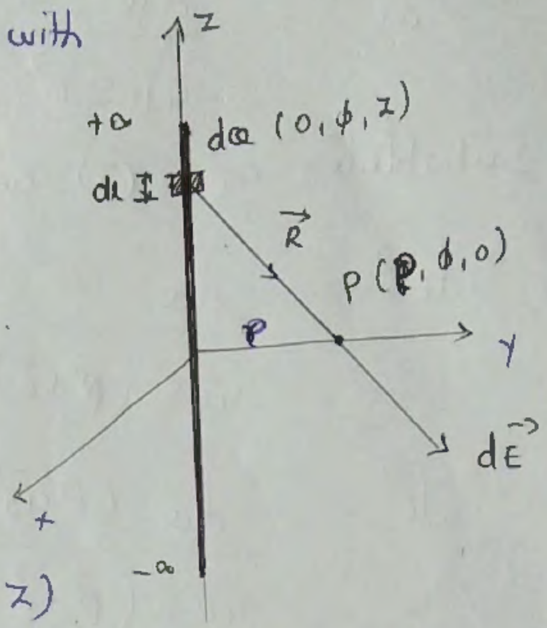
Electric field intensity of an infinite line charge



Consider a charge with uniform density (ρ_l) is distributed along an infinite straight line.

Now consider a small segment of line charge dl at $(0, \phi, z)$

$$\therefore dQ = \rho_l \, dl \qquad \text{But } dl = dz$$

$$= \rho_l \, dz \rightarrow (1)$$

Suppose we have to calculate $\vec{E}$ at point P in the xy plane and coordinates of P be $(\rho, \phi, 0)$

$$\therefore \vec{R} = \vec{P} - \vec{dQ}$$

$$= (\rho \vec{a_\rho} + \phi \vec{a_\phi}) - (\phi \vec{a_\phi} + z \vec{a_z})$$

$$\vec{R} = \rho \vec{a_\rho} - z \vec{a_z} \rightarrow (2)$$

$$|\vec{R}| = R = \sqrt{\rho^2 + z^2} \rightarrow (3)$$

$$\vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\rho \vec{a_\rho} - z \vec{a_z}}{\sqrt{\rho^2 + z^2}} \rightarrow (4)$$

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According to definition of $\vec{E}$,

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon R^2}\vec{a_R} \rightarrow (5)$$

Substitute eq's (3) and (4) in to (5)

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon(\rho^2+z^2)} \cdot \frac{\rho\vec{a_\rho} - z\vec{a_z}}{(\rho^2+z^2)^{1/2}}$$

$$d\vec{E} = \frac{dQ\,(\rho\vec{a_\rho} - z\vec{a_z})}{4\pi\varepsilon(\rho^2+z^2)^{3/2}} \rightarrow (6)$$

We are considering a small charge dQ present on the ~~negative~~ +z axis, this same charge is present on the −z axis. So the z-components in eq (6) will get cancelled.

$$\therefore d\vec{E} = \frac{dQ\,\rho\vec{a_\rho}}{4\pi\varepsilon(\rho^2+z^2)^{3/2}} \rightarrow (7)$$

Now, the total $\vec{E}$ at point P is obtained by taking line integral of eq (7)

$$\vec{E} = \int_{-\infty}^{\infty} d\vec{E}$$

$$\therefore \vec{E} = \int_{-\infty}^{\infty} \frac{dQ\,\rho\vec{a_\rho}}{4\pi\varepsilon(\rho^2+z^2)} \rightarrow (8)$$

Substitute eq (1) in to (8)

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$$\vec{E} = \int_{-\infty}^{\infty} \frac{\rho_l \, \rho \, dz \; \vec{a_\rho}}{4\pi \varepsilon (\rho^2 + z^2)^{3/2}} \rightarrow (9)$$

$$\therefore \tan\theta = \frac{z}{\rho}$$

Put $z = \rho \tan\theta$

$$dz = \rho \sec^2\theta \, d\theta$$

limits of integration will changes as follows

$$z \rightarrow \infty \; ; \; \theta \rightarrow \pi/2$$

$$z \rightarrow -\infty \; ; \; \theta \rightarrow -\pi/2$$

Thus eq (9) becomes

$$\vec{E} = \int_{-\pi/2}^{\pi/2} \frac{\rho_l \, \rho \, (\rho \sec^2\theta \, d\theta) \; \vec{a_\rho}}{4\pi\varepsilon \; (\rho^2 + \rho^2 \tan^2\theta)^{3/2}} \rightarrow (10)$$

But
$$(\rho^2 + \rho^2 \tan^2\theta)^{3/2} = (\rho^2)^{3/2} (1 + \tan^2\theta)^{3/2}$$
$$= \rho^3 (\sec^2\theta)^{3/2}$$
$$= \rho^3 \sec^3\theta$$

Thus eq (10) becomes

$$\vec{E} = \int_{-\pi/2}^{\pi/2} \frac{\rho_l \, \cancel{\rho^2} \, \sec^{\cancel{2}}\theta \, d\theta \; \vec{a_\rho}}{4\pi\varepsilon \; \rho^{\cancel{3}} \sec^{\cancel{3}}\theta}$$

$$= \frac{\rho_l \, \vec{a_\rho}}{4\pi\varepsilon\rho} \int_{-\pi/2}^{\pi/2} \frac{1}{\sec\theta} \, d\theta$$

$$= \frac{\rho_l \, \vec{a_\rho}}{4\pi\varepsilon\rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{\rho_l}{4\pi\varepsilon\rho} \left(-\sin\theta\right)_{-\pi/2}^{\pi/2}$$

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$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon\rho}\,\vec{a}_\rho\left[\sin\pi/2 - \sin(-\pi/2)\right]$$

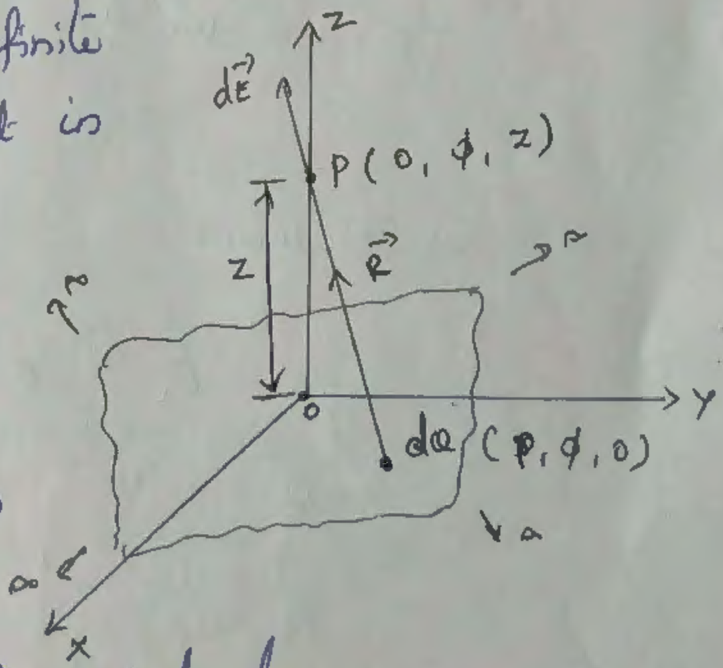
$$= \frac{\rho_L}{4\pi\varepsilon\rho}\,\vec{a}_\rho\left[1-(-1)\right]$$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\,\vec{a}_\rho} \quad V/m \quad \rightarrow (11)$$

Eq (11) gives electric field intensity at point $P(\rho, \phi, z)$ due to infinite line charge.

<u>Electric field intensity ($\vec{E}$) due to infinite uniformly charge sheet :-</u>

Consider an infinite surface charge sheet in the xy plane.



In cylindrical coordinate system, the coordinates of P and dQ are $(0, \phi, z)$ and $(\rho, \phi, 0)$ respectively.

Now we will create a small charge dQ with $(\rho, \phi, 0)$.

$$\therefore dQ = \rho_s\, ds \rightarrow (1)$$

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According to the definition of $\vec{E}$,

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon R^2}\,\vec{a_R} \rightarrow (2)$$

$$\vec{R} = \vec{P} - d\vec{Q}$$

$$= (\phi\,\vec{a_\phi} + z\,\vec{a_z}) - (\rho\,\vec{a_\rho} + \phi\,\vec{a_\phi})$$

$$= -\rho\,\vec{a_\rho} + z\,\vec{a_z} \longrightarrow (3)$$

$$|\vec{R}| = R = \sqrt{\rho^2 + z^2} \longrightarrow (4)$$

$$\vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{-\rho\,\vec{a_\rho} + z\,\vec{a_z}}{\sqrt{\rho^2 + z^2}} \rightarrow (5)$$

Substitute eq's (4) and (5) in to (2)

$$d\vec{E} = \frac{dQ}{4\pi\varepsilon(\rho^2+z^2)} \cdot \frac{-\rho\,\vec{a_\rho} + z\,\vec{a_z}}{(\rho^2+z^2)^{3/2}}$$

$$= \frac{dQ\,(-\rho\,\vec{a_\rho} + z\,\vec{a_z})}{4\pi\varepsilon(\rho^2+z^2)^{3/2}} \rightarrow (6)$$

But, charge sheet is placed in the xy plane, every horizontal components will get cancelled.

$\vec{a_\rho} \rightarrow$ horizontal components

$a_z \rightarrow$ vertical components

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$$\therefore \; d\vec{E} = \frac{dQ \; z \; \vec{a_z}}{4\pi\varepsilon (\rho^2+z^2)^{3/2}}$$

But $dQ = \rho_s \, ds_z$ $\qquad ds_z = \rho \, d\rho \, d\phi$

$= \rho_s \, \rho \, d\rho \, d\phi .$

$$\therefore \; d\vec{E} = \frac{\rho_s \, \rho \, d\rho \, d\phi \; z \; \vec{a_z}}{4\pi\varepsilon (\rho^2+z^2)^{3/2}} \longrightarrow (7)$$

Now, total $\vec{E}$ is obtained by taking integration of eq (7).

$$\int_S d\vec{E} = \vec{E}$$

$$\therefore \; \vec{E} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho_s \, \rho \, z \, d\rho \, d\phi}{4\pi\varepsilon (\rho^2+z^2)^{3/2}} \vec{a_z}$$

$$= \frac{\rho_s \, z \; \vec{a_z}}{4\pi\varepsilon} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{\rho \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}}$$

Put $\rho^2 + z^2 = t$

$2\rho \, d\rho = dt$

$d\rho = \dfrac{dt}{2\rho}$

<u>limits of integration</u>:

$\rho \to 0 \; ; \; t \to z^2$

$\rho \to \infty \; ; \; t \to \infty$

$$\therefore \; \vec{E} = \frac{\rho_s \, z \; \vec{a_z}}{4\pi\varepsilon} \int_{\phi=0}^{2\pi} \int_{t=z^2}^{\infty} \frac{\rho \, dt \, d\phi}{2\rho \cdot t^{3/2}}$$

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$$\vec{E} = \frac{\rho_s z \vec{a_z}}{8\pi\varepsilon} (\phi)_0^{2\pi} \int_{\rho = z^2}^{\infty} t^{-3/2} \, dt$$

$$= \frac{\rho_s z \vec{a_z}}{\cancel{8}\cancel{\pi}\varepsilon} \times 2\cancel{\pi} \times \left[\frac{t^{-3/2+1}}{-\frac{3}{2}+1} \right]_{z^2}^{\infty}$$

4

$$= \frac{\rho_s z}{\cancel{4}\varepsilon} \vec{a_z} \times (-\cancel{2}) \times \left(t^{-1/2} \right)_{z^2}^{\infty}$$

2

$$= \frac{\rho_s \cancel{z}}{2\varepsilon} \times (-1) \times (0 - \cancel{z}^{-1})$$

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon} \vec{a_z}}$$ V/m $\Rightarrow$ Point P is is above xy plane.

$$\boxed{\vec{E} = -\frac{\rho_s}{2\varepsilon} \vec{a_z}}$$ V/m $\Rightarrow$ point P is is below xy plane.

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UNIT - III MAGNETOSTATICS

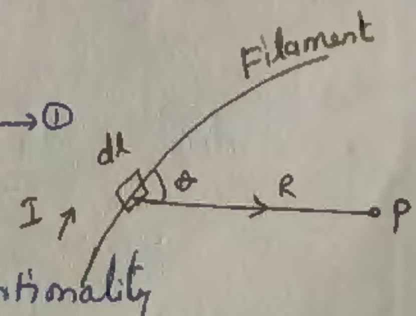
Lorentz force equation, Law of magnetic monopoles, Ampere's law, vector magnetic potential, Biot - savart law and applications, Magnetic field intensity and idea of relative permeability, Magnetic circuits, Behaviour of magnetic materials, Boundary conditions, inductance and Inductors, Magnetic energy, Magnetic forces and torques.

* **Biot - Savart law :**

Statement :-

" The Biot Savart law states that at any point P, the magnitude of magnetic field intensity produced by the differential element is proportional to the product of current, magnitude of differential length and sine of angle between the filament and the line connecting differential length to point P. and also inversely proportional to the square of distance from the differential element to the point P."

$$\therefore \quad dH \propto \frac{I\, dl \sin\theta}{R^2} \quad \rightarrow ①$$



Let k be the constant of proportionality

$$dH = k\,\frac{I\, dl \sin\theta}{R^2} \quad \rightarrow ②$$

But $k = \frac{1}{4\pi}$

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$$\therefore \boxed{dH = \frac{I\,dL \sin\theta}{4\pi R^2}} \rightarrow ③$$

In vector form,

$$d\vec{H} = \frac{I\,d\vec{L} \sin\theta}{4\pi R^2} \vec{a}_R$$

But

$$\vec{A} \times \vec{B} = |A||B| \sin\theta$$

$$\therefore d\vec{L} \times \vec{a}_R = |d\vec{L}||\vec{a}_R| \sin\theta$$

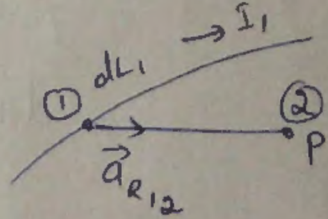
$$\boxed{d\vec{H} = \frac{I\,d\vec{L} \times \vec{a}_R}{4\pi R^2}} \rightarrow ④$$

Now total MFI at point P is obtained by integration of eq ④,

$$\int d\vec{H} = \vec{H}$$

$$\therefore \boxed{\vec{H} = \oint \frac{I\,d\vec{L} \times \vec{a}_R}{4\pi R^2}} \text{ A/m.} \rightarrow ⑤$$

* If we consider current element is located at point 1 and we have to calculate MFI at point 2



$$d\vec{H_2} = \frac{I_1\,d\vec{L_1} \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} \rightarrow ⑥$$

$$\therefore \vec{H_2} = \oint \frac{I_1\,d\vec{L_1} \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} \text{ A/m} \rightarrow ⑦$$

where

$$\vec{a}_{R_{12}} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|}$$

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Similarly, $I\,d\vec{L} = \vec{k}\,ds$ $\qquad$ $I\,d\vec{L} = \vec{J}\,dv$

$$\therefore \vec{H} = \int_S \frac{\vec{k} \times \vec{a_R}}{4\pi R^2} \rightarrow (8) \qquad \int_V \frac{\vec{J} \times \vec{a_R}}{4\pi R^2} \rightarrow (9)$$

* Magnetic field intensity :- $(\vec{H})$

Magnetic field intensity is defined as the force experienced by a unit north pole of one weber strength

$$\therefore \vec{H} = \frac{\vec{F}}{\phi} \text{ in N/wb (or) A/m (or) AT/m}$$

ϕ - Magnetic flux in weber

* Magnetic flux density :- $(\vec{B})$

Magnetic flux density is defined as the flux passing per unit area

$$\therefore \vec{B} = \frac{\phi}{S} \text{ Wb/m}^2 \text{ (or) Tesla}$$

* Magnetic flux :- (ϕ)

The total magnetic flux passing through the surface is

$$\phi = \oint_S \vec{B} \cdot \vec{ds}$$

* Relation between $\vec{B}$ and $\vec{H}$

$$\vec{B} = \mu \vec{H} \qquad \therefore \mu = \mu_0 \mu_r$$

For free space, $\mu_r = 1$

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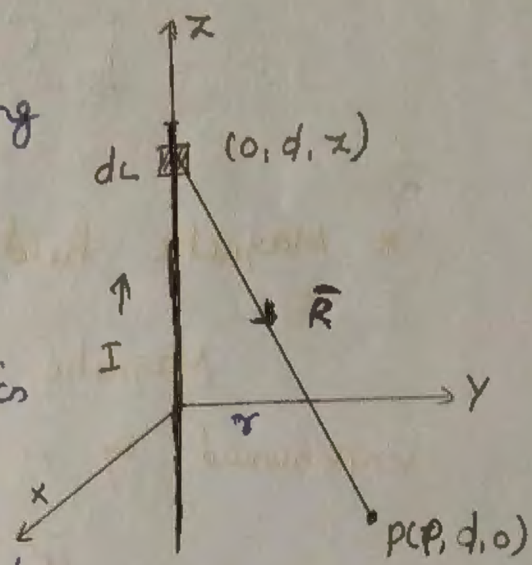
* Magnetic field intensity due to infinite wire carrying current I.

Consider an infinitely long straight filament or wire is placed along z-axis



Now consider P is placed in the xy plane at P(ρ, ϕ, 0) and small differential element $\vec{dL}$ along z axis at $\vec{dL}(0, \phi, z)$

From Fig, $d\vec{L} = dz$

$$\therefore I\vec{dL} = I\,dz\,\vec{a_z} \rightarrow ①$$

$$\vec{R} = \vec{P} - \vec{dL} = (\rho - 0)\,\vec{a_r} + (\phi - \phi)\,\vec{a_\phi} + (0 - z)\,\vec{a_z}$$

$$\vec{R} = \rho\,\vec{a_\rho} - z\,\vec{a_z} \rightarrow ②$$

$$\therefore |\vec{R}| = \sqrt{\rho^2 + z^2} \rightarrow ③$$

$$\vec{a_R} = \frac{\rho\vec{a_\rho} - z\vec{a_z}}{\sqrt{\rho^2 + z^2}} \rightarrow ④$$

According to Biot-Savart law,

$$\vec{dH} = \frac{I\,\vec{dL} \times \vec{a_R}}{4\pi R^2} \rightarrow ⑤$$

Substitute eq's ①, ③ and ④ in to eq ⑤,

$$\vec{dH} = \frac{I\,dz\,\vec{a_z}}{4\pi(\rho^2 + z^2)} \times \frac{\rho\vec{a_\rho} - z\vec{a_z}}{(\rho^2 + z^2)^{1/2}}$$

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$$d\vec{H} = \frac{I\,dz\,\vec{a_z}}{4\pi(\rho^2+z^2)^{3/2}} \times (\rho\,\vec{a_\rho} - z\,\vec{a_z})$$

But $\vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$ and $\vec{a_z} \times \vec{a_z} = 0$

$$\therefore\ d\vec{H} = \frac{I\,dz\,\rho\,\vec{a_\phi}}{4\pi(\rho^2+z^2)^{3/2}} \rightarrow ⑥$$

The total MFI is given by,

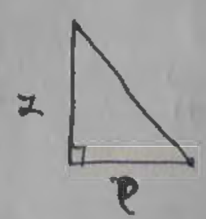
$$\int d\vec{H} = \vec{H}$$

By taking integration of eq ⑥,

$$\vec{H} = \int_{-\infty}^{\infty} \frac{I\,dz\,\rho\,\vec{a_\phi}}{4\pi(\rho^2+z^2)^{3/2}}$$

$$= \frac{I\rho}{4\pi}\vec{a_\phi} \int_{-\infty}^{\infty} \frac{dz}{(\rho^2+z^2)^{3/2}} \rightarrow ⑦$$

Put $\tan\theta = \frac{z}{r}$



$$z = \rho\tan\theta$$

$$dz = \rho\sec^2\theta\,d\theta$$

$$\therefore\ (\rho^2+z^2)^{3/2} = (\rho^2+\rho^2\tan^2\theta)^{3/2}$$

$$= (\rho^2)^{3/2}(1+\tan^2\theta)^{3/2}$$

$$= \rho^3(\sec^2\theta)^{3/2}$$

$$= \rho^3\sec^3\theta$$

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The limits of integration

$z \to \infty \quad , \quad \theta \to \pi/2$

$z \to -\infty \quad \theta \to -\pi/2$

$$\therefore \vec{H} = \frac{I \cancel{\rho} \, \vec{a}_\phi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{\cancel{\rho} \sec^{\cancel{2}}\theta \, d\theta}{\rho^{\cancel{2}} \sec^{\cancel{3}}\theta}$$

$$= \frac{I \cancel{\rho} \, \hat{a}_\phi}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\cancel{\rho}^{\cancel{2}} \sec\theta} \, d\theta$$

$$= \frac{I \, \vec{a}_\phi}{4\pi\rho} \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{I}{4\pi\rho} \hat{a}_\phi \left(\sin\theta\right)_{-\pi/2}^{\pi/2}$$

$$= \frac{I \, \hat{a}_\phi}{4\pi\rho} \left[\sin \pi/2 - \sin(-\pi/2)\right]$$

$$= \frac{I}{4\pi\rho} \hat{a}_\phi \left(1 - (-1)\right)$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \vec{a}_\phi} \quad A/m$$

But $\vec{B} = \mu_0 \vec{H}$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi} \quad Wb/m^2$$

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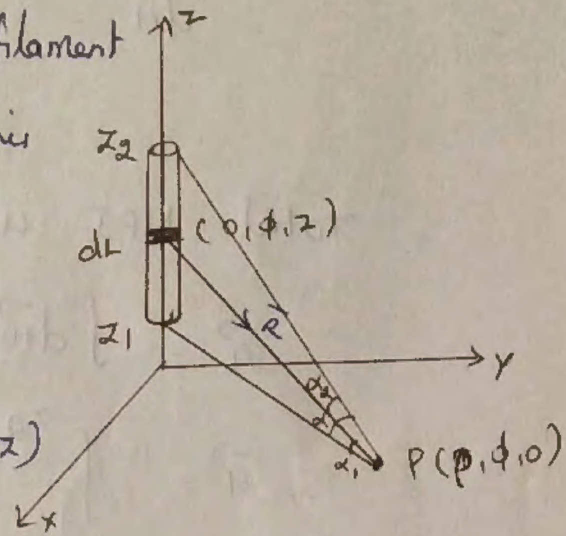
* **Magnetic field intensity due to finite length current filament (or) conductor :-**

Consider a finite length filament carrying current I along z axis



$\therefore dL = dz$

$$I\vec{dL} = I\,dz\,\vec{a_z} \rightarrow ①$$

dL having coordinates $(0,\phi,z)$ and P is in xy plane at $P(\rho,\phi,0)$.

$$\therefore \vec{R} = \vec{P} - \vec{dL} = (\rho-0)\vec{a_r} + (\phi-\phi)\vec{a_\phi} + (0-z)\vec{a_z}$$

$$\vec{R} = \rho\vec{a_r} - z\vec{a_z} \rightarrow ②$$

$$\therefore |\vec{R}| = R = \sqrt{\rho^2+z^2} \rightarrow ③$$

$$\vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\rho\vec{a_r} - z\vec{a_z}}{\sqrt{\rho^2+z^2}} \rightarrow ④$$

According to Biot-Savart law

$$\vec{dH} = \frac{I\vec{dL}\times\vec{a_R}}{4\pi R^2} \rightarrow ⑤$$

Substitute eqn ①, ③ and ④ in to eq ⑤,

$$\vec{dH} = \frac{I\,dz\,\vec{a_z}}{4\pi(\rho^2+z^2)} \times \frac{\rho\vec{a_r} - z\vec{a_z}}{(\rho^2+z^2)^{1/2}}$$

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But $\vec{a_z} \times \vec{a_z} = 0$ and $\vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$

$$\therefore \vec{dH} = \frac{I\, dz\, \rho\, \vec{a_\phi}}{4\pi(\rho^2+z^2)^{3/2}} \rightarrow (6)$$

Total MFI is given by

$$\vec{H} = \int \vec{dH}$$

$$\therefore \vec{H} = \int_{z_1}^{z_2} \frac{I\, dz\, \rho\, \vec{a_\phi}}{4\pi(\rho^2+z^2)^{3/2}}$$

Put $\tan\alpha = \frac{z}{\rho}$; $z = \rho \tan\alpha$

$$dz = \rho \sec^2\alpha \, d\alpha$$

$$\therefore (\rho^2+z^2)^{3/2} = (\rho^2 + \rho^2\tan^2\alpha)^{3/2}$$

$$= (\rho^2)^{3/2}(1+\tan^2\alpha)^{3/2}$$

$$= \rho^3 \sec^3\alpha .$$

Limits:

At $z \to z_1$; $z_1 = \rho\tan\alpha_1$; $\alpha_1 = \tan^{-1}(z_1/\rho)$

$z \to z_2$; $z_2 = \rho\tan\alpha_2$; $\alpha_2 = \tan^{-1}(z_2/\rho)$

$$\therefore \vec{H} = \frac{I}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho(\rho\sec^2\alpha)\, d\alpha}{\rho^3\sec^3\alpha}\vec{a_\phi}$$

$$= \frac{I\,\vec{a_\phi}}{4\pi\rho}\int_{\alpha_1}^{\alpha_2}\frac{1}{\sec\alpha}\, d\alpha$$

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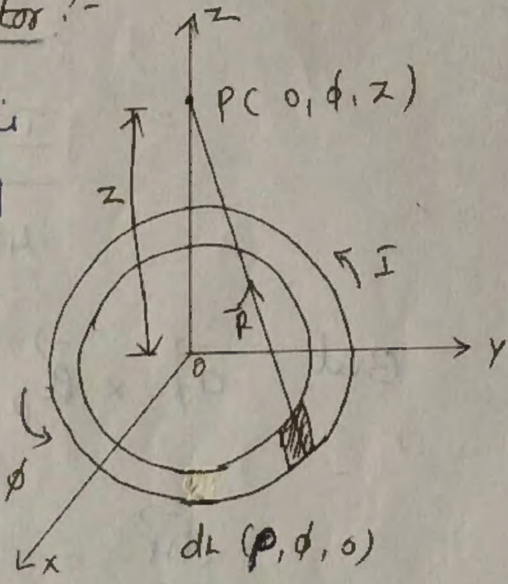
$$\vec{H} = \frac{I}{4\pi\rho}\hat{a}_\phi \int_{\alpha_1}^{\alpha_2} \cos\alpha \, d\alpha = \frac{I}{4\pi\rho}\vec{a}_\phi \left(\sin\alpha\right)_{\alpha_1}^{\alpha_2}$$

$$\therefore \boxed{\vec{H} = \frac{I}{4\pi\rho}\left(\sin\alpha_2 - \sin\alpha_1\right)\vec{a}_\phi} \quad Wb/m^2.$$

* Magnetic field intensity on the axis of a Circular loop (or) conductor :-



Consider a circular loop is centered at origin and placed in the xy plane.

The point is located at the z-axis $P(0, \phi, z)$ and coordinates of dL are $(\rho, \phi, 0)$.

$$\therefore d\vec{L} = \rho \, d\phi \, \vec{a}_\phi \quad (\text{in cylindrical coordinates})$$

$$I\,d\vec{L} = I\rho \, d\phi \, \vec{a}_\phi \rightarrow (1)$$

$$\vec{R} = \vec{P} - d\vec{L} = (0-\rho)\vec{a}_\rho + (\phi-\phi)\vec{a}_\phi + (z-0)\vec{a}_z$$

$$\therefore \vec{R} = -\rho\vec{a}_\rho + z\,\vec{a}_z \rightarrow (2)$$

$$|\vec{R}| = R = \sqrt{\rho^2+z^2} \rightarrow (3)$$

$$\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{-\rho\vec{a}_\rho + z\,\vec{a}_z}{\sqrt{\rho^2+z^2}} \rightarrow (4)$$

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According to Biot-Savart law

$$d\vec{H} = \frac{I\, d\vec{L} \times \vec{a}_R}{4\pi R^2} \rightarrow (5)$$

Substitute eq's (1), (3) & (4) in to eq (5)

$$d\vec{H} = \frac{I \rho\, d\phi\, \vec{a}_\phi}{4\pi(\rho^2+z^2)} \times \frac{-\rho\vec{a}_\rho + z\vec{a}_z}{(\rho^2+z^2)^{1/2}}$$

$$= \frac{I\rho\, d\phi\, \vec{a}_\phi}{4\pi(\rho^2+z^2)^{3/2}} \times (-\rho\vec{a}_\rho + z\vec{a}_z)$$

But $\vec{a}_\phi \times \vec{a}_\rho = -\vec{a}_z$ and $\vec{a}_\phi \times \vec{a}_z = \vec{a}_\rho$

$$\therefore d\vec{H} = \frac{I\rho^2\, d\phi\, \vec{a}_z + I\rho z\, \vec{a}_\rho\, d\phi}{4\pi(\rho^2+z^2)^{3/2}} \rightarrow (6)$$

Now, circular loop is placed along xy plane ($z=0$)

$\therefore$ (i.e. horizontal plane components will get cancelled) ($\vec{a}_\rho = 0$)

$$\therefore \int d\vec{H} = \int_{\phi=0}^{2\pi} \frac{I\rho^2\, d\phi\, \vec{a}_z}{4\pi(\rho^2+z^2)^{3/2}}$$

$$\vec{H} = \frac{I\rho^2\, \vec{a}_z}{4\pi(\rho^2+z^2)^{3/2}} \int_0^{2\pi} d\phi$$

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$$\vec{H} = \frac{I \rho^2 \vec{a_z}}{4\pi (r^2+z^2)^{3/2}} \times 2\pi$$

$$\therefore \boxed{\vec{H} = \frac{I \rho^2 \vec{a_z}}{2(\rho^2+z^2)^{3/2}}} \; Wb/m^2 \rightarrow 7$$

If Point P is placed at origin, then $z=0$, thus eq 7 becomes

$$\vec{H} = \frac{I \rho^2 \vec{a_z}}{2\rho^3}$$

$$\therefore \boxed{\vec{H} = \frac{I}{2\rho} \vec{a_z}} \; Wb/m^2$$

---

* **Ampere's circuital law :-**

Statement :-

"Ampere's Circuital law states the line integral of magnetic field intensity ($\vec{H}$) about any closed path is equal to the direct current enclosed by that path."

$$\therefore \boxed{\oint_l \vec{H} \cdot d\vec{l} = I} \rightarrow 1$$

Proof :-

consider a long straight conductor carrying current I. Magnetic field intensity of conductor is

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$$\vec{H} = \frac{I}{2\pi R}$$, R - Radius of circular path

But $\vec{B} = \mu_0 \vec{H}$

$$\therefore \vec{B} = \frac{\mu_0 I}{2\pi R} \rightarrow (2)$$

Taking line integral of eq (2)

$$\oint \vec{B} \cdot \vec{dl} = \frac{\mu_0 I}{2\pi R} \oint \vec{dl}$$

But $B = B \cdot \vec{a_\phi}$; $\vec{dl} = dl\, \vec{a_\phi}$; $\vec{a_\phi} \cdot \vec{a_\phi} = 1$

$$\oint \vec{B} \cdot \vec{dl} = \frac{\mu_0 I}{2\pi R} \times 2\pi R$$

$$\oint \frac{B}{\mu_0} \cdot \vec{dl} = I$$ But $B = \mu_0 H$

$$\therefore \boxed{\oint \vec{H} \cdot \vec{dl} = I} \rightarrow (3)$$

Applications of Ampere's circuital law:

1. Curl
2. Stoke's theorem
3. MFI due to long straight conductor
4. MFI due to coaxial cable

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* Scalar magnetic potential

V_m - Scalar magnetic potential in volts

$$\therefore \vec{H} = -\nabla V_m \quad \text{if } \vec{J} = 0$$

The scalar magnetic potential is also applicable in case of permanent magnets.

$$\therefore \nabla^2 V_m = 0 \quad \text{for } \vec{J} = 0 \rightarrow \text{satisfies Laplace's equation}$$

* vector magnetic potential :-

The vector magnetic potential may be used in regions where the current density is zero or non-zero.

It is denoted by $\vec{A}$ in Wb/m.

$\vec{A}$ satisfies,

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \rightarrow (1)$$

From point form Gauss's law

$$\nabla \cdot \vec{B} = 0 \rightarrow (2)$$

Comparing eq's (1) and (2)

$$\boxed{\vec{B} = \nabla \times \vec{A}} \rightarrow (3)$$

According to point form of Ampere's law

$$\nabla \times \vec{H} = \vec{J} \rightarrow (4)$$

But $B = \mu_0 \vec{H} \quad \therefore \vec{H} = \dfrac{\vec{B}}{\mu_0}$

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$$\therefore \nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} \rightarrow (4)$$

Using eq (3)

$$\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J} \rightarrow (5)$$

Using vector identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Hence eq (5) becomes

$$\mu_0 \vec{J} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\therefore \boxed{\vec{J} = \frac{1}{\mu_0}\left(\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right)} \rightarrow (6)$$

Now, consider a differential element $d\vec{l}$ and current I passing through it.

Using Biot - Savart law

$$\vec{A} = \oint_l \frac{\mu I}{4\pi R} dl \quad Wb/m \quad \text{(line charge)}$$

$$\vec{A} = \int_S \frac{\mu \vec{k}\, ds}{4\pi R} \quad Wb/m \quad \text{(Surface charge)}$$

$$I\,dl = \vec{k}\, ds$$

$$\vec{A} = \oint_V \frac{\mu \vec{J}\, dv}{4\pi R} \quad Wb/m \quad \text{(Volume charge)}$$

$$I\,dl = \vec{J}\, dv$$

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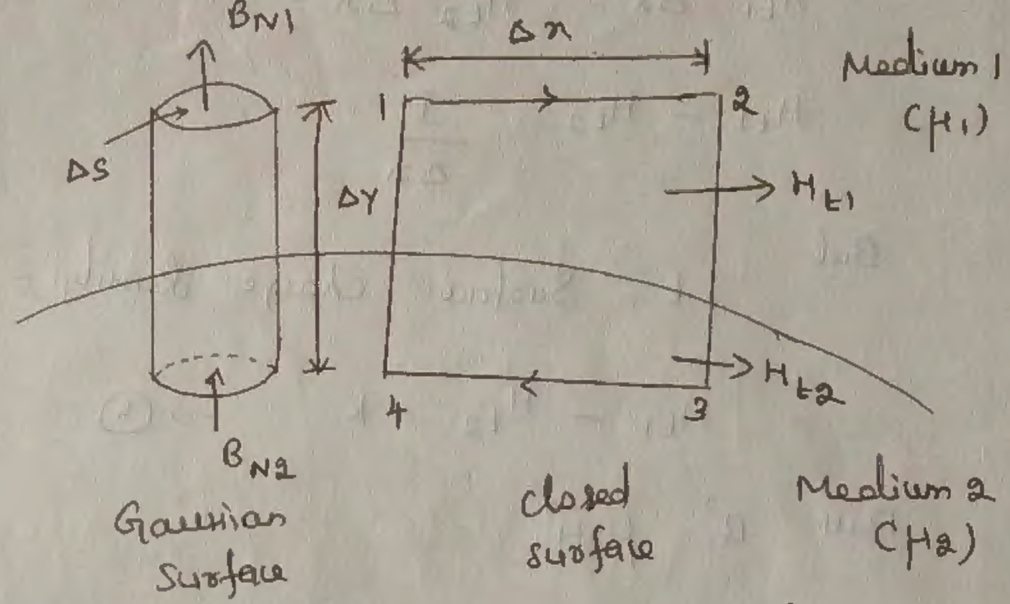
* Magnetic boundary conditions :-



To find tangential components of $\vec{B}$ and $\vec{H}$:-

According to Ampere's Circuital law

$$\oint_l \vec{H} \cdot d\vec{l} = I \rightarrow (1)$$

Consider a closed path 1-2-3-4-1,

$$\int_1^2 \vec{H} \cdot d\vec{l} + \int_2^3 \vec{H} \cdot d\vec{l} + \int_3^4 \vec{H} \cdot d\vec{l} + \int_4^1 \vec{H} \cdot d\vec{l} = I \rightarrow (2)$$

If we consider the height of Surface $\Delta y \rightarrow 0$

$$\therefore \int_2^3 \vec{H} \cdot d\vec{l} = \int_4^1 \vec{H} \cdot d\vec{l} = 0$$

Hence equation (2) becomes

$$\int_1^2 \vec{H} \cdot d\vec{l} + \int_3^4 \vec{H} \cdot d\vec{l} = I \rightarrow (3)$$

For path 1-2, component of $\vec{H}$ is $\vec{H}_{t1}$ and for path 3-4, component of $\vec{H}$ is $-\vec{H}_{t2}$.

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$\therefore H_{t1} \int d\vec{l} - H_{t2} \int dl = I$

$\therefore \int dl = \Delta x$

$H_{t1} \Delta x - H_{t2} \Delta x = I$

$H_{t1} - H_{t2} = \dfrac{I}{\Delta x}$

But

$k = \text{Surface charge density} = \dfrac{I}{\Delta x}$

$$\therefore \boxed{H_{t1} - H_{t2} = k} \rightarrow 4$$

But $\vec{B} = \mu \vec{H}$

$B_{t1} = \mu_1 H_{t1}$ & $B_{t2} = \mu_2 H_{t2}$

$$\therefore \boxed{\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = k} \rightarrow 5$$

<u>To find normal components of $\vec{B}$ and $\vec{H}$.</u>

According to Gauss's law

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \rightarrow 6$$

Consider a cylindrical Gaussian surface,

$$\int_{Top} \vec{B} \cdot d\vec{s} + \int_{Bottom} \vec{B} \cdot d\vec{s} + \int_{Lateral} \vec{B} \cdot d\vec{s} = 0 \rightarrow 7$$

If we consider height $\Delta y \rightarrow 0$, $\therefore \int_{Lateral} \vec{B} \cdot d\vec{s} = 0$

$\therefore$ eq 7 becomes

$$\int_{Top} \vec{B} \cdot d\vec{s} + \int_{Bottom} \vec{B} \cdot d\vec{s} = 0 \rightarrow 8$$

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For the top surface, the component of $\vec{B}$ is $+B_{N1}$ and for bottom surface the component of $\vec{B}$ is $-B_{N2}$.

$$\therefore \quad B_{N1} \int d\vec{s} - B_{N2} \int ds = 0 \qquad \therefore \int ds = \Delta S$$

$$\Delta S \left(B_{N1} - B_{N2} \right) = 0$$

$$\therefore \quad B_{N1} - B_{N2} = 0$$

$$\boxed{B_{N1} = B_{N2}} \rightarrow (9)$$

But

$$\vec{B} = \mu \vec{H}$$

$$\vec{B}_{N1} = \mu_1 \vec{H}_{N1} \quad ; \quad \vec{B}_{N2} = \mu_2 \vec{H}_{N2}$$

$$\therefore \quad \mu_1 \vec{H}_{N1} = \mu_2 \vec{H}_{N2}$$

$$\boxed{\frac{\vec{H}_{N1}}{\vec{H}_{N2}} = \frac{\mu_2}{\mu_1}} \rightarrow (10)$$

---

* <u>Lorentz's force equation for moving charge</u>:

For an electric field,

$$\vec{F} = Q \vec{E} \rightarrow (1)$$

For an magnetic field,

$$\vec{F} \propto Q \, |\vec{V}| \, |\vec{B}| \sin\theta$$

$\vec{V}$ = velocity vector (m/s) ; $\vec{B}$ = Magnetic flux density (wb/m^2)

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$\therefore \vec{F} = Q\, |\vec{v}|\, |\vec{B}| \sin\theta$

using cross product rule

$\vec{v} \times \vec{B} = |\vec{v}|\, |\vec{B}| \sin\theta$

$\therefore \vec{F} = Q(\vec{v} \times \vec{B}) \rightarrow ②$

Lorentz's force is the combined form of electric and magnetic fields. By adding eq's ① & ②

$\therefore \vec{F} = Q\vec{E} + Q(\vec{v} \times \vec{B})$

$$\boxed{\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})} \rightarrow ③$$

Eq ③ is called as Lorentz force equation.

Force on a differential current element (or) Force on a wire carrying current I placed in a magnetic field.

Consider a conductor carrying current I placed in steady magnetic field.

The force on the differential current element is given by

$$d\vec{F} = dQ(\vec{v} \times \vec{B}) \rightarrow ①$$

w.k.t

$$\rho_v = \frac{Q}{V}$$

$$Q = \rho_v . V \rightarrow ②$$

$$\therefore dQ = \rho_v\, dv \rightarrow ③$$

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Substitute eq ③ into ①,

$$d\vec{F} = \rho_v \, dv \, (\vec{v} \times \vec{B}) \rightarrow ④$$

But $\vec{J} = \rho_v \vec{v}$

$$\therefore \; d\vec{F} = \vec{J} \times \vec{B} \; dv$$

$$\boxed{d\vec{F} = \vec{J} dv \times \vec{B}} \rightarrow ⑤$$

Now, $\vec{J} = \frac{I}{ds}$ and $dv = ds \, d\vec{L}$

$$\therefore \; \vec{J} dv = \frac{I}{ds} ds \, d\vec{L}$$

$$\vec{J} dv = I d\vec{L}$$

Hence eq ⑤ becomes

$$\therefore \; d\vec{F} = I d\vec{L} \times \vec{B}$$

But

$$\int d\vec{F} = \vec{F}$$

$$\vec{F} = \oint I \, d\vec{L} \times \vec{B}$$

$$\therefore \; \boxed{\vec{F} = I \oint d\vec{L} \times \vec{B} = -I \oint \vec{B} \times d\vec{L}}$$

(line charge)

$$\vec{F} = \oint_V (\vec{J} \times \vec{B}) dv \quad \text{(volume charge)}$$

$$\vec{F} = \oint_S (\vec{K} \times \vec{B}) d\vec{s} \quad \text{(surface charge)}$$

For straight conductor,

$$\vec{F} = I\vec{L} \times \vec{B} = ILB \sin\theta \; \vec{a_n}$$

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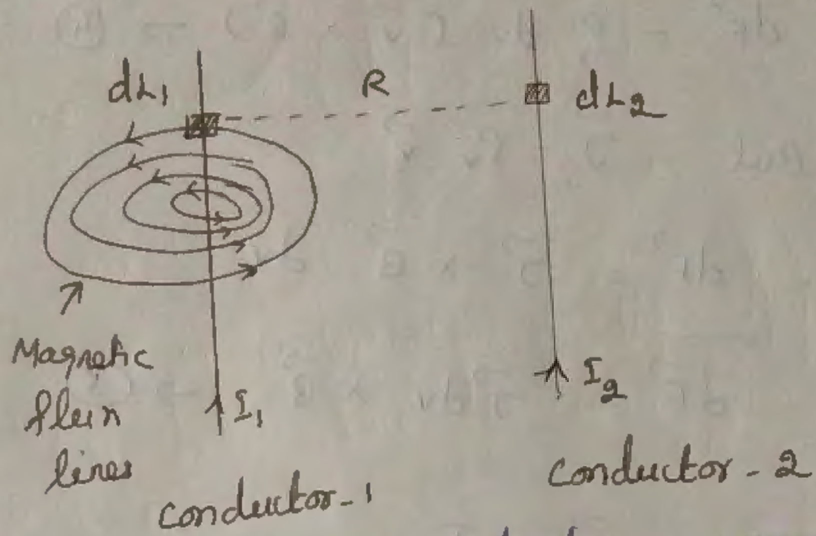
Force between differential current elements:



Consider two straight conductors carrying currents I_1 & I_2 respectively.

The differential force on the differential current element-2,

$$d\vec{F_2} = I_2\, d\vec{L_2} \times \vec{B_1} \rightarrow (1)$$

using Biot-savart law,

$$\vec{H_1} = \oint \frac{I\, d\vec{L_1} \times \vec{a_R}}{4\pi R^2} \rightarrow (2)$$

But $B = \mu_0 \vec{H} \Rightarrow \vec{B_1} = \mu_0 \vec{H_1}$

$$\therefore \vec{B_1} = \oint \frac{\mu_0 I_1\, d\vec{L_1} \times \vec{a_R}}{4\pi R^2} \rightarrow (3)$$

Substitute eq (3) in to eq (1),

$$d\vec{F_2} = I_2\, d\vec{L_2} \times \oint \frac{\mu_0 I_1\, d\vec{L_1} \times \vec{a_R}}{4\pi R^2}$$

$$= I_2\, d\vec{L_2} \times \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{L_1} \times \vec{a_R}}{R^2}$$

$$\boxed{\therefore d\vec{F_2} = \frac{\mu_0 I_1 I_2}{4\pi} \left[d\vec{L_2} \times \oint \frac{d\vec{L_1} \times \vec{a_R}}{R^2} \right]} \rightarrow (4)$$

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$$\therefore \vec{F_2} = \frac{\mu_0 I_1 I_2}{4\pi}\left\{\oint\left[d\vec{L_2} \times \oint \frac{d\vec{L_1} \times \vec{a_R}}{R^2}\right]\right\} \quad (21)$$

$\hookrightarrow$ (5)

Eq (5) gives the force on the second conductor.

Similarly, the first conductor force is

$$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi}\left\{\oint d\vec{L_1} \times \oint \frac{d\vec{L_1} \times \vec{a_R}}{R^2}\right]\right\} \rightarrow (6)$$

Force and Torque on a Closed circuit :-

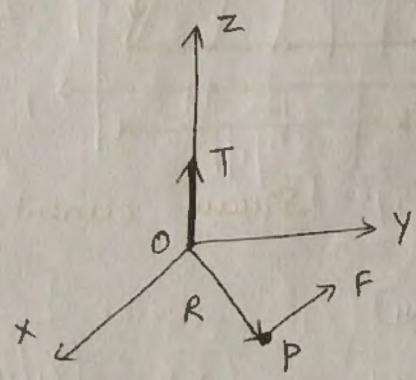

Fig - a) Torque about origin
$T = R \times F$

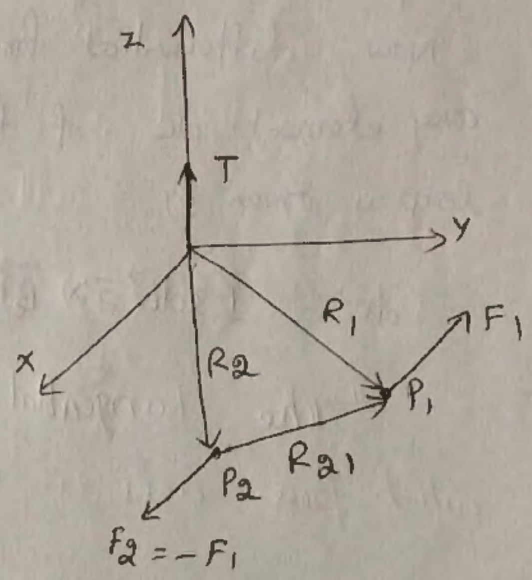

Fig - b) If $F_2 = -F_1$, $T = R_{21} \times F_1$

The direction of vector torque (T) is normal to both force (F) and lever arm (R).

$$\therefore T = \vec{R} \times \vec{F} = RF \sin\theta \rightarrow (1)$$

From Fig b)

$$T = R_1 \times F_1 + R_2 \times F_2 \rightarrow (2)$$

But $F_2 = -F_1$ $\therefore F_1 + F_2 = 0$

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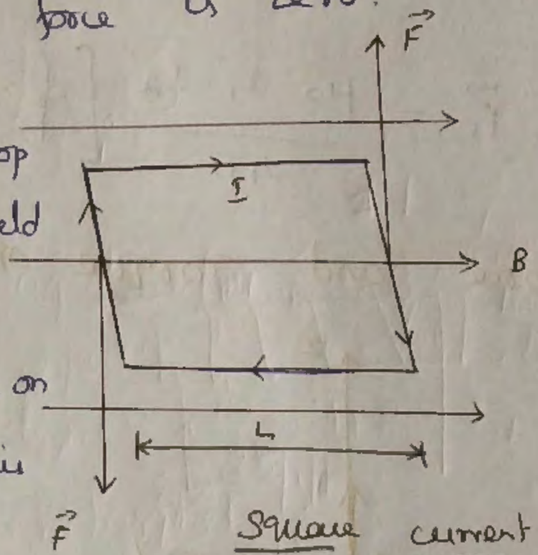
and $T = R_1 \times F_1 + R_2 \times (-F_1)$

$= (R_1 - R_2) \times F_1$

$= R_{21} \times F_1 \rightarrow (3)$

Hence, torque is independant of the choice of origin, i.e, Total force is zero.



Square current loop

Consider a square loop placed in the magnetic field $\vec{B}$.

Now, differential force on any element, dL of this loop is given by

$d\vec{F} = I\, d\vec{L} \times \vec{B} \rightarrow (4)$

"The tangential times the radial distance at which force acts is called as torque".

$\therefore F_t = 2ILB \sin\theta \rightarrow (5)$

$T = F_t \cdot \frac{L}{2} = 2ILB \sin\theta \cdot \frac{L}{2}$

$\therefore T = IL^2 B \sin\theta \rightarrow (6)$

But L^2 = Area of loop = A

$\therefore \boxed{T = IAB \sin\theta} \rightarrow (7)$

But, The product of current and area is called magnetic moment of the loop (m)

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$\therefore m = IA \rightarrow (8)$

$$\boxed{T = mB \sin\theta} \rightarrow (9)$$

In vector form

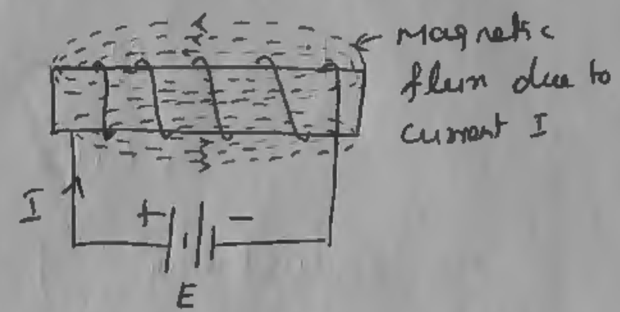
$$\vec{m} = m\vec{a_n}$$

$$\therefore \vec{T} = mB \sin\theta \, \vec{a_n} \qquad \because \vec{m} \times \vec{B} = mB \sin\theta \, \vec{a_n}$$

$$\boxed{\therefore \vec{T} = \vec{m} \times \vec{B}}$$

Inductance :-

* Self inductance :-



When a coil with N turns, carrying current I, the flux is produced by it.

'The ratio of total flux linkage to the current producing that flux is called inductance denoted by L".

It is measured in henry (H).

$$\therefore \boxed{L = \frac{N\phi}{I} \text{ Henry (H).}}$$

* Mutual inductance .

Mutual inductance between two coils is defined as the ratio of flux linkage of one coil to current in other coil.

The mutual inductance depends on magnetic interaction between the two currents.

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$$\therefore M_{12} = \frac{N_2 \phi_{12}}{I_1}$$

and $$M_{21} = \frac{N_1 \phi_{21}}{I_2}$$

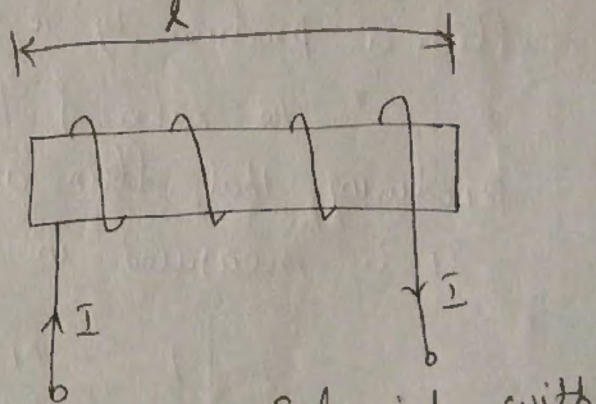
* coefficient of coupling :

The fraction of the total flux produced by one coil linking a second coil is called coefficient of coupling (k).

$$k = \frac{M}{\sqrt{L_1 L_2}} \qquad \therefore M = k\sqrt{L_1 L_2}$$

* Inductance of Solenoid :

Consider a solenoid of N turns and carrying current I. ~~the~~ length of the solenoid be 'l' and cross sectional area of 'A'.



Solenoid with 'N' turns

Now, the field intensity inside the solenoid is

$$H = \frac{NI}{l} \longrightarrow ①$$

The total flux linkage is

$$N\phi = N \cdot B \cdot A \qquad \therefore B = \phi/A$$

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But $B = \mu H$

$$\therefore N\phi = N\mu H A \rightarrow (2)$$

From eq (1),

$$N\phi = \mu . N . \frac{N . I}{l} . A$$

$$\therefore N\phi = \frac{\mu N^2 I A}{l} \rightarrow (3)$$

The inductance of solenoid is

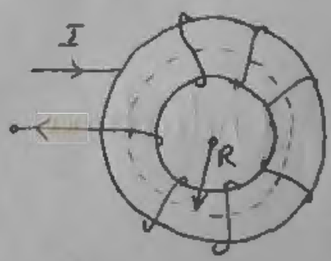
$$L = \frac{\text{Total flux linkage}}{\text{Total current}} = \frac{N\phi}{I}$$

From eq (3),

$$L = \frac{\mu N^2 I A}{l \, I}$$

$$\therefore \boxed{L = \frac{\mu N^2 A}{l}} \text{ Henry}$$

* **Inductance of Toroid**



Consider a toroidal ring with N turns and carrying current I.

R – Radius of the Toroid

The Magnetic flux density inside a toroidal ring is given by

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$$B = \frac{\mu N I}{2\pi R}$$

$\therefore$ Total flux linkage, $N\phi = N B . A$

$$N\phi = N . \frac{\mu N I}{2\pi R} . A = \frac{\mu N^2 I A}{2\pi R}$$

$\therefore$ The total inductance of toroid is

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$= \frac{\mu N^2 I A}{2\pi R . I}$$

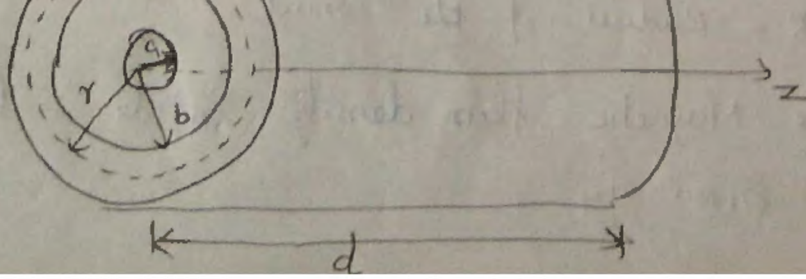
$$\therefore \boxed{L = \frac{\mu N^2 A}{2\pi R}} \text{ Henry}$$

But $A = \cancel{2}\pi r^2$

$$\therefore L = \frac{\mu N^2}{2\cancel{\pi} R} \times \cancel{\pi} r^2$$

$$\boxed{L = \frac{\mu N^2 r^2}{2R}} \text{ Henry}$$

Inductance of a coaxial cable



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Consider a coaxial cable with inner conductor radius 'a' and outer conductor radius 'b'.

Let the current through the coaxial cable be I.

The MFI at any point between inner and outer conductor is

$$\vec{H} = \frac{I}{2\pi r}\vec{a_\phi} \quad , a < r < b \rightarrow (1)$$

But $\vec{B} = \mu \vec{H}$

$$\therefore \vec{B} = \frac{\mu I}{2\pi r}\vec{a_\phi} \longrightarrow (2)$$

Now, the total magnetic flux is given by

$$\phi = \oint_S \vec{B} \cdot \vec{ds} \longrightarrow (3)$$

But $\vec{ds_\phi} = dr\, dz\, \vec{a_\phi}$ $\quad \therefore \vec{a_\phi} \cdot \vec{a_\phi} = 1$

$$\therefore \phi = \int_{z=0}^{d}\int_{r=a}^{b}\left(\frac{\mu I}{2\pi r}\vec{a_\phi}\right) \cdot \left(dr\, dz\, \vec{a_\phi}\right)$$

$$= \frac{\mu I}{2\pi}\int_{z=0}^{d}\int_{r=a}^{b}\left(\frac{1}{r}dr\right)(dz)$$

$$= \frac{\mu I}{2\pi}(\ln r)_a^b \, (z)_0^d$$

$$= \frac{\mu I d}{2\pi}(\ln b - \ln a)$$

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$$\phi = \frac{\mu I d}{2\pi} \ln(b/a) \rightarrow (4)$$

∴ The inductance of coaxial cable is

$$L = \frac{\text{Total flux linkage}}{\text{Total current}} = \frac{\phi}{I}$$

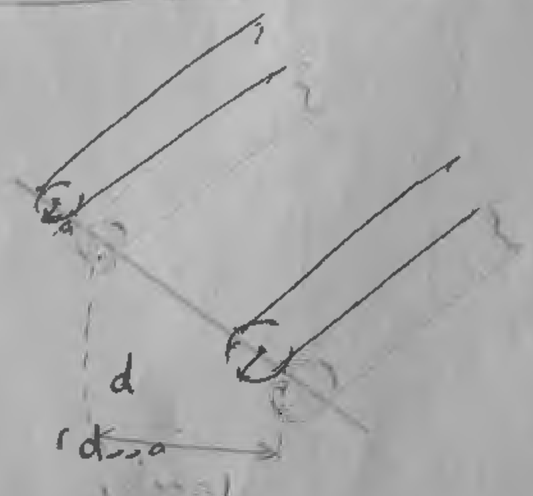
$$= \frac{\mu I d}{2\pi I} \ln(b/a)$$

$$\boxed{L = \frac{\mu d}{2\pi} \ln(b/a) \text{ Henry}}$$

The inductance of a coaxial cable per unit length is

$$\boxed{\frac{L}{d} = \frac{\mu}{2\pi} \ln(b/a) \quad H/m}$$

Inductance of a transmission line



Consider a transmission line made up of two long parallel conductors of radius 'a' carrying currents in opposite direction with a separation distance of d.

Energy stored in conductor is given by

$$W_m = \frac{1}{2} L I^2 \rightarrow (1)$$

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$$L = \frac{2W_m}{I^2}$$

But $W_m = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\,dv = \int \frac{B^2}{2\mu}\,dv \rightarrow (2)$

For inner conductor

$$L_{in} = \frac{2W_m}{I^2} \rightarrow (3)$$

Substitute eq (3) into (2)

$dv = \rho\, d\rho\, d\phi\, dz$

$B = \frac{\mu I \rho}{2\pi a^2}$

$$L_{in} = \frac{\cancel{2}}{I^2}\int_V \frac{B^2}{\cancel{2}\mu}\,dv$$

$$= \frac{1}{I^2\mu}\int_v \frac{\mu^2 I^2 \rho^2}{4\pi^2 a^4}(\rho\, d\rho\, d\phi\, dz)$$

$$= \frac{\mu}{4\pi^2 a^4}\int_0^l dz \int_0^{2\pi} d\phi \int_0^a \rho^3 d\rho$$

$$= \frac{\mu}{4\pi^2 a^4}(z)_0^l\,(\phi)_0^{2\pi}\left(\frac{\rho^4}{4}\right)_0^a$$

$$L_{in} = \frac{\mu}{4\pi^2 a^4}\, l \times 2\pi \times \frac{a^4}{4}$$

$$L_{in} = \frac{\mu l}{8\pi}$$

Similarly, for outer conductor

$$L_{out} = \frac{2}{I^2}\int \frac{B^2}{2\mu}\,dv$$

$dv = \rho\, d\rho\, d\phi\, dz \quad ; \quad B = \frac{\mu I}{2\pi\rho}$

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$$L_{out} = \frac{2}{2\mu I^2} \int_v \frac{\mu^2 I^2}{4\pi^2 \rho^2} \rho \, d\rho \, d\phi \, dz$$

$$= \frac{\mu}{4\pi^2} \int_0^l dz \int_0^{2\pi} d\phi \int_a^{d-a} \frac{1}{\rho} d\rho$$

$$= \frac{\mu}{4\pi^2} (l)(2\pi)(\ln \rho)_a^{d-a}$$

$$= \frac{\mu l}{2\pi} \ln\left(\frac{d-a}{a}\right)$$

Hence, total inductance for transmission line is

$$L = 2[L_{in} + L_{out}]$$

$$= 2\frac{\mu l}{8\pi} + 2\frac{\mu l}{2\pi} \ln\left(\frac{d-a}{a}\right)$$

$$= \frac{\mu l}{4\pi} + \frac{\mu l}{\pi} \ln\left(\frac{d-a}{a}\right)$$

$$\boxed{L = \frac{\mu l}{\pi}\left[\frac{1}{4} + \ln\left(\frac{d-a}{a}\right)\right]} \text{ Henry}$$

For $d >> a$, $\therefore d - a = d$

$$\boxed{L = \frac{\mu l}{\pi}\left[\frac{1}{4} + \ln(d/a)\right]} \text{ Henry}$$

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- Energy Stored in magnetic field :

In order to establish a current in the circuit, certain amount of work is required. This work represents energy transferred to the circuit.

Consider an example of toroid,

$$H = \frac{NI}{l} \rightarrow ①$$

Now induced voltage is given by

$$V = -N \frac{d\phi}{dt} \longrightarrow ②$$

But $\phi = B.A$

$$\therefore V = -NA \frac{dB}{dt} \rightarrow ③$$

The workdone required to establish a current I is

$$W = -\int_0^{t_1} VI\, dt \rightarrow ④$$

Substitute eq ③ into eq ④,

$$W = \int_0^{t_1} NAI \frac{dB}{dt} . dt$$

$$= \int_0^{t_1} \frac{NI}{l} . A.l \frac{dB}{dt} dt$$

From eq ①,

$$W = \int_0^{t_1} H l A \frac{dB}{dt} dt$$

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But $B = \mu H$

$$\therefore W = \int_0^{t_1} \mu H l A \frac{dH}{dt} dt$$

Put $t = H$; $dt = dH$

and at $t = t_1$, $H = H_1$

$$\therefore W = \int_0^{H_1} \mu H l A \, dH$$

$$= \mu l A \left(\frac{H^2}{2}\right)_0^{H_1}$$

$$\therefore \boxed{W = l A \left(\frac{\mu H_1^2}{2}\right)} \rightarrow (5) \Rightarrow \text{"Total energy in magnetic field"}$$

Now length (l) × Area (A) = Volume (V)

$$\therefore \boxed{\frac{W}{V} = \frac{\mu H_1^2}{2}} \rightarrow (6) \Rightarrow \text{"Energy density in magnetic field"}$$

In a linear medium,

$$W_m = \frac{1}{2} \int \vec{B} \cdot \vec{H} \, dv \quad (\text{or})$$

$$W_m = \frac{1}{2} \int \mu H^2 \, dv \quad (\text{or})$$

$$W_m = \frac{1}{2} \int \frac{B^2}{\mu} \, dv$$

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* The nature of magnetic materials :-

Magnetic materials are classified on the basis of presence of magnetic dipole moment in materials.

→ orbital magnetic dipole moment
→ Electron spin magnetic moment
→ Nuclear spin magnetic moment.

On the basis of magnetic behaviour the magnetic materials are classified as

i) Diamagnetic :-

The magnetic materials in which the orbital magnetic moment and electron spin magnetic moment cancel each other making net permanent magnetic momentum of each atom zero called diamagnetic materials.

Properties :-

* $\chi_m < 0$, $\mu_r \leq 1$
* Permanent dipoles are absent.
* Temperature independent.
* Linear magnetic materials
* When placed inside a magnetic field, magnetic lines of force are repeled.

Ex: copper, Lead, Silicon, Diamond

ii) Paramagnetic :-

Each electron in an orbit has orbital magnetic moment and spin magnetic moment. They are do not cancel each other.

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Properties :-

* $\chi_m > 0$, $\mu_r \geq 1$
* Linear magnetic materials
* Temperature dependent
* Spin alignment is random ↓ → ↓ ←
* Ex: Air, Tungsten, Platinum
* When placed inside a magnetic field, it attracts lines of force.

iii) Ferromagnetic :- "Large no. of magnetic moment lined in parallel called domains".

Properties :

* $\chi_m >> 0$, $\mu_r >> 1$
* Spin alignment is parallel ↑↑↑↑
* It posses large permanent dipole moment
* Non-linear.
* They lose their properties when temperature is raised.
* Ex: Iron, Nickel, cobalt

iv) Antiferromagnetic materials :-

Material in which dipole moment of adjacent atoms line up in antiparallel are called antiferromagnetic.

Properties :

* Net magnetic moment is zero
* Temperature dependent
* Spin alignment is antiparallel ↑↓↑↓
* Ex : oxide, chloride and sulphides.

v) Ferri magnetic : Properties :

* Spin alignment is antiparallel of different magnitude ↑↓↑↓↑
* It posses net magnetic moment
* Ferrite are special case of ferrimagnetic

Ex: Nickel ferrite, Nickel - zinc - Ferrite

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vi) Super paramagnetic :-

These materials are the combination of ferromagnetic particle within a non ferromagnetic matrix.

Ex: Magnetic tape.

* Magnetization and permeability :-

Magnetization is defined as the magnetic dipole moment per unit volume.

$$M = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i=1}^{n\Delta V} m_i \rightarrow (1)$$

where m_i - magnetic dipole moment $= I_b \, ds$.

According to Ampere's circuital law, $I = \oint \vec{H} \cdot d\vec{l}$ $\rightarrow (2)$

$$I_b = \oint M \cdot dl \Rightarrow \text{Bounded current} \rightarrow (3)$$

But $I_T = I_b + I \rightarrow (4)$ I - Free current

$$\therefore I_T = \oint \frac{B}{\mu_0} dl \rightarrow (5)$$

Substitute eq's (5) & (3) in to (4)

$$I = I_T - I_b$$

$$= \oint \frac{B}{\mu_0} dl - \oint M \cdot dl$$

$$I = \oint \left(\frac{B}{\mu_0} - M\right) dl \rightarrow (6)$$

Comparing eq's (2) and (6),

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$$H = \frac{B}{\mu_0} - M \quad ; \quad \boxed{M = \frac{B}{\mu_0} - H} \rightarrow (7)$$

$$B = \mu_0 (H+M) \rightarrow (8)$$

Permeability :

$$M = \chi_m H \rightarrow (9) \qquad \chi_m - \text{magnetic susceptibility}$$

Substitute eq (9) in to (8)

$$B = \mu_0 (H + \chi_m H)$$

$$= \mu_0 H (1 + \chi_m)$$

But $\mu_r = 1 + \chi_m$

$$\therefore B = \mu_0 \mu_r H$$

But $\mu = \mu_0 \mu_r$

$$\therefore \boxed{B = \mu H}$$

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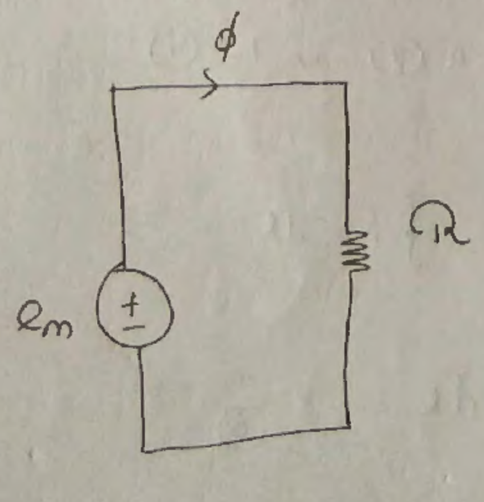
* **The magnetic circuit :**



Magnetic circuit

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A single magnetic lines of flux or all parallel magnetic lines of flux may be considered as magnetic circuit.

$\therefore$ Magnetomotive force (m.m.f) $= e_m$

$$e_m = NI = \oint \vec{H} \cdot d\vec{l} \rightarrow (1)$$

S.I unit of m.m.f is Ampere (A)

Reluctance (R): It is the ratio of m.m.f to the total flux

$$\therefore R = \frac{e_m}{\phi} = \frac{l}{\mu S} = \frac{d}{\mu S} \rightarrow (2)$$

For magnetic circuit,

$$\vec{B} = \mu \vec{H} \rightarrow (3)$$

and other magnetostatics equations are

$$\nabla \cdot \vec{B} = 0 \rightarrow (4)$$

$$\nabla \times \vec{H} = \vec{J} \rightarrow (5)$$

$$H = -\nabla V_m \rightarrow (6)$$

Relationship between m.m.f and MFI is

$$V_m = \int_A^B \vec{H} \cdot d\vec{l} \rightarrow (7)$$

and also $I = \oint_S \vec{J} \cdot d\vec{s} \rightarrow (8)$

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The total magnetic flux is

$$\phi = \oint_s \vec{B} \cdot d\vec{s} \rightarrow (9)$$

Permeance : It is measured in Henry (H)

$$P = \frac{\mu S}{l} = \frac{\mu S}{d} \rightarrow (10)$$

Kirchoff's flux law :-

" The total magnetic flux arriving at any junction in a magnetic circuit is equal to the total magnetic flux leaving that junction".

$$\therefore \Sigma \phi = 0 \rightarrow (11)$$

Kirchoff's m.m.f. law :-

" The resultant m.m.f. around a closed magnetic circuit is equal to the algebric sum of product of flux and reluctance of each part of the closed circuit".

$$\therefore \Sigma \text{m.m.f} = \Sigma \phi R \rightarrow (12)$$

(or)

$$\Sigma \text{m.m.f} = \Sigma . H . l . \rightarrow (13)$$

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Faraday's law, Displacement current and Maxwell Ampere law, Maxwell's equations, Potential functions, Electromagnetic boundary conditions, wave equations and solutions, Time - harmonic fields.

* <u>Faraday's law for electromagnetic induction:</u>

<u>Statement:</u>

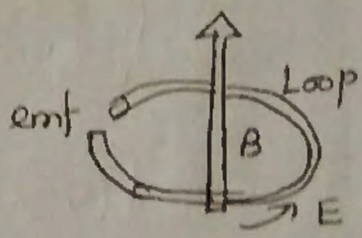
"The electromotive force around a closed path is equal to the negative of rate of change of magnetic flux enclosed by that path".

$$emf = -\frac{d\phi}{dt} \quad (\text{volts}) \rightarrow (1)$$

where ϕ = Total flux (wb)

$\frac{d\phi}{dt}$ = Time rate of change of magnetic flux

Consider open circuited loop as shown in figure. Let us assume that the gap in open circuited loop is negligible.



<u>open circuited loop</u>

Then the emf induced in the loop is equal to the emf producing electric field (E) integrated all the way around the loop.

$$\therefore emf = \oint_L \vec{E} \cdot \vec{dl} \rightarrow (2)$$

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Now, the total magnetic flux is equal to the surface integral of the magnetic flux density.

$$\therefore \phi = \oint_S \vec{B} \cdot \vec{ds} \rightarrow ③$$

Substitute eq ③ in eqn ①,

$$emf = -\frac{d}{dt} \int_s \vec{B} \cdot \vec{ds} \rightarrow ④$$

Now, equating eq ② and eq ④,

$$\oint_L \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_s \vec{B} \cdot \vec{ds} \rightarrow ⑤$$

Eq ⑤ is called as Maxwell's equation derived from Faraday's law.

Now, By applying Stoke's theorem to the L.H.S. of eq ⑤,

$$\oint_L \vec{E} \cdot \vec{dl} = \oint_S (\nabla \times \vec{E}) \cdot \vec{ds}$$

$$\therefore \oint_S (\nabla \times \vec{E}) \cdot \vec{ds} = -\int \frac{\partial \vec{B}}{\partial t} \vec{ds}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow ⑥$$

Eq ⑥ is called point form of Maxwell's equation obtained from Faraday's law.

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* Transformer emf and Motional emf :

When an e.m.f is induced in a stationary closed path due to time varying $\vec{B}$ field, the emf is called statically induced e.mf or transformer emf.

When an emf is induced in the time varying closed path due to a static $\bar{B}$ field, the e.m.f is called dynamically induced emf or motional e.m.f

* Different conditions related to Faraday's law :

Condition 1 :- A stationary closed path in a time varying $\bar{B}$ field - Transformer emf.

In this case, the loop is not varying with respect to time but the magnetic flux density ($\bar{B}$) varies with time.

$$\therefore \quad emf = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} \quad \rightarrow (1)$$

Condition 2 : A moving closed path in static $\vec{B}$ field - Motional emf.

In this case, the magnetic field moving with velocity ($\bar{V}$) and loop is moving is called motional induction

$$\therefore \quad \vec{F} = Q\vec{V} \times \vec{B} \quad \rightarrow (2)$$

$$\vec{E} = \frac{\vec{F}}{Q} = \vec{V} \times \vec{B} \quad \rightarrow (3)$$

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$$\therefore \ emf = -\oint_l \vec{E} \cdot d\vec{l}$$

$$emf = -\oint_l (\vec{V} \times \vec{B}) \cdot d\vec{l} \longrightarrow (4)$$

Condition III :- Moving closed path in a time varying $\vec{B}$ field.

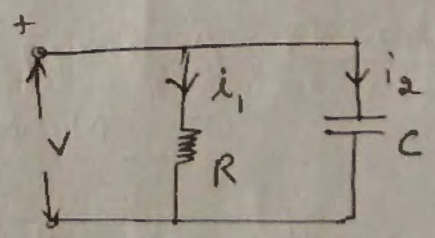
In this case, the loop is moving with velocity $(\vec{V})$ and the magnetic field is also changing with respect to time

$\therefore$ Total induced emf = Transformer emf + motional emf

$$\boxed{emf = \oint_l (\vec{V} \times \vec{B}) \cdot d\vec{l} - \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}} \rightarrow (5)$$

---

* Displacement current and current density :-



Elements of R :

$\sigma, \vec{E}, \vec{J_c}$

Elements of C :

$\sigma, \vec{E}, \varepsilon, \vec{J_D}$

Consider a parallel R-C circuit shown in Figure

The current through resistor is due to actual motion of charges.

$$i_1 = \frac{V}{R} = i_c \rightarrow (1)$$

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This current is called conduction current as the current is flowing because of actual motion of charges. It is denote by i_c and conduction current density is J_c.

$$\boxed{\therefore J_c = \frac{i_c}{A} = \sigma \vec{E}} \rightarrow (2)$$

Now, the current through the capacitor is

$$i_2 = C \frac{dv}{dt} \rightarrow (3)$$

Let the two plates of area A are separated by distance d with dielectric permittivity ε between the plates

$$\therefore i_2 = \frac{\varepsilon A}{d} \frac{dv}{dt} \rightarrow (4)$$

This current is called displacement current denoted by i_D and density of this current is displacement current density J_D.

$$\therefore E = \frac{V}{d} \Rightarrow V = \vec{E} . d \rightarrow (5)$$

Substitute eq (5) in to (4)

$$i_2 = i_D = \frac{\varepsilon A}{d} \frac{d}{dt}(E d)$$

$$\therefore i_D = \varepsilon A \frac{d\vec{E}}{dt} \rightarrow (6)$$

and also,

$$J_D = \frac{i_D}{A} = \frac{\varepsilon A}{A} \frac{d\vec{E}}{dt}$$

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$$\vec{J}_D = \frac{d}{dt}(\varepsilon \vec{E})$$

But $\vec{D} = \varepsilon \vec{E}$

$$\therefore \boxed{\vec{J}_D = \frac{\partial \vec{D}}{\partial t}} \rightarrow (7)$$

Hence, the total current density is given by

$$\vec{J} = \vec{J}_c + \vec{J}_D$$

$$= \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \rightarrow (8)$$

Using Ampere's circuital law,

$$\nabla \times \vec{H} = \vec{J}$$

$$\therefore \boxed{\nabla \times \vec{H} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}}$$

* **Maxwell's Equations :-**

* Differential form and integral form of Maxwell's equations for time varying fields :-

A) Maxwell's Equation I :- This equation is derived from Ampere's law

Integral form :

" The line integral of magnetic field intensity over a closed path is equal to the surface integration of current density".

$$\therefore \oint_l \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}$$

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Proof:

According to Ampere's law

$$\oint_l \vec{H} \cdot d\vec{l} = I = \int_s \vec{J} \cdot d\vec{s} \rightarrow (1)$$

Now, current through a conductor of resistor R is

$$I_c = \frac{V}{R} = \sigma \vec{E} A$$

$$\therefore J_c = \frac{I_c}{A} = \sigma \vec{E} = \vec{J} \rightarrow (2)$$

Current through a capacitor is,

$$C = Q/V$$

$$I_D = \frac{dQ}{dt} \quad ; \; Q = CV$$

$$= C \frac{dv}{dt} \quad ; \; C = \frac{\varepsilon A}{d} \text{ and } V = \vec{E} \cdot d$$

$$\therefore I_D = \frac{\varepsilon A}{d} \frac{d\vec{E}d}{dt} = \varepsilon A \frac{d\vec{E}}{dt}$$

But $\vec{D} = \varepsilon \vec{E}$

$$\therefore I_D = A \frac{\partial \vec{D}}{\partial t}$$

$$\vec{J_D} = \frac{I_D}{A} = \frac{\partial \vec{D}}{\partial t} \rightarrow (3)$$

From eq (1),

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s (\vec{J_c} + \vec{J_D}) \cdot d\vec{s}$$

$$\boxed{\oint_l \vec{H} \cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}} \rightarrow (4)$$

Eq (4) is Maxwell's equation I in integral form.

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Point form :-

"The curl of magnetic field intensity is equal to the current density".

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Proof:

By applying Stoke's theorem,

$$\oint_l \vec{H} \cdot d\vec{l} = \int_s (\nabla \times \vec{H}) \cdot d\vec{s} \rightarrow 5$$

Comparing eq's 4 and 5,

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}} \rightarrow 6$$

Eq 6 is called Maxwell's equation I in differential or point form.

B, **Maxwell's Equation II :-**

Integral form :- This equation is derived from Faraday's law

"The line integral of electric field intensity is equal to the surface integral of rate of decrease of magnetic flux linkage in the circuit".

$$\boxed{\oint_l \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}} \rightarrow 7$$

Proof: According to Faraday's law

$$emf = -\frac{d\phi}{dt} \rightarrow 8$$

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But. $\phi = \int_s \vec{B} \cdot d\vec{s} \rightarrow (9)$

Substitute eq (9) in to (8),

$$e.m.f = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \rightarrow (10)$$

consider open circuited loop,

$$e.m.f = \oint_l \vec{E} \cdot d\vec{l} \rightarrow (11)$$

Comparing eq's (10) & (11),

$$\int_l \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \rightarrow (12)$$

Equation (12) is called Maxwell's equation II in integral form

Point form (or) differential form:-

By applying Stoke's theorem

$$\oint_l \vec{E} \cdot d\vec{l} = \oint_s (\nabla \times \vec{E}) \cdot d\vec{s} \rightarrow (13)$$

comparing eq's (12) and (13),

$$\int \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (14)$$

" The curl of electric field intensity is equal to the rate of decrease of magnetic flux density".

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \rightarrow (15)$$

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c) Maxwell's Equation III :

Integral form : "This equation is derived from Gauss's law in electric field"

"The surface integral of electric flux density over a closed surface is equal to the volume integral of volume charge density".

$$\therefore \boxed{\oint_s \vec{D} \cdot d\vec{s} = \int_v \rho_v \, dv} \rightarrow (16)$$

Proof :

According to Gauss's law, the electric flux passing through any closed surface is equal to the charge enclosed by that surface.

$$\therefore \psi = Q \rightarrow (17)$$

But $\psi = \oint_s \vec{D} \cdot d\vec{s}$

and $Q = \int_v \rho_v \, dv$

$$\therefore \oint_s \vec{D} \cdot d\vec{s} = \int_v \rho_v \, dv \rightarrow (18)$$

Eq (18) is called Maxwell's equation III in integral form.

Point form :

"The electric flux per unit volume is equal to the volume charge density".

$$\therefore \boxed{\nabla \cdot \vec{D} = \rho_v} \rightarrow (19)$$

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Proof :-

By applying divergence theorem

$$\oint_S \vec{D}\cdot d\vec{s} = \int_V \nabla\cdot\vec{D}\, dv \rightarrow (20)$$

Comparing eq's (18) and (20),

$$\boxed{\nabla\cdot\vec{D} = \rho_v} \rightarrow (21)$$

Eq (21) is called Maxwell's equation III in point form.

D, Maxwell's Equation IV :-

Integral form :- "This equation is derived from Gauss's law in magnetic field.

" The net magnetic flux entering and leaving the closed surface is same."

$$\therefore \boxed{\oint_S \vec{B}\cdot d\vec{s} = 0}$$

Proof :-

According to Gauss's law in magnetic field, total magnetic flux through any closed surface is equal to zero.

$$\therefore \phi = 0$$

$$\boxed{\int_S \vec{B}\cdot d\vec{s} = 0} \rightarrow (22)$$

Eq (22) is called Maxwell's equation IV in integral form.

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Point form:

"For a static magnetic field, total outgoing flux is zero

$$\therefore \boxed{\nabla \cdot \vec{B} = 0}$$

Proof:

Applying divergence theorem

$$\oint_S \vec{B} \cdot \vec{ds} = \int_v \nabla \cdot \vec{B}\, dv \rightarrow (23)$$

Comparing eq's (22) and (23)

$$\int_v \nabla \cdot \vec{B} = 0$$

$$\therefore \boxed{\nabla \cdot \vec{B} = 0} \rightarrow (24)$$

Eq (24) is called Maxwell's equation IV in point form.

Maxwell's equations in static fields:

	Integral form	Point form
I.	$\oint_l \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot \vec{ds}$	$\nabla \times \vec{H} = \vec{J}$
II	$\int_l \vec{E} \cdot \vec{dl} = 0$	$\nabla \times \vec{E} = 0$
III	$\oint_S \vec{D} \cdot \vec{ds} = \int_v \rho_v\, dv$	$\nabla \cdot \vec{D} = \rho_v$
IV	$\oint_S \vec{B} \cdot \vec{ds} = 0$	$\nabla \cdot \vec{B} = 0$

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Maxwell's equation for free space :

For free space , $\vec{J} = 0$ and $\rho_v = 0$

(Also valid for good dielectrics)

	Integral form	Point form
I	$\oint_l \vec{H} \cdot d\vec{l} = \int_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$	$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$
II	$\oint_l \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
III	$\oint_s \vec{D} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{D} = 0$
IV	$\oint_s \vec{B} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{B} = 0$

Maxwell's equation for good conductors :

$\vec{J} >> \frac{\partial \vec{D}}{\partial t}$ and $\rho_v = 0$

	Integral form	Point form
I	$\oint_l \vec{H} \cdot d\vec{l} = \int_s \vec{J} \cdot d\vec{s}$	$\nabla \times \vec{H} = \vec{J}$
II	$\oint_l \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
III	$\oint_s \vec{D} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{D} = 0$
IV	$\oint_s \vec{B} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{B} = 0$

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Wave equations for free space :-

The free space is a non-conducting medium

$\therefore \quad J_c = 0 \quad$ and $\quad \rho_v = 0$

Recall Maxwell's equation in point form

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (1)$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

$$\nabla \cdot \vec{D} = 0 \rightarrow (3)$$

$$\nabla \cdot \vec{B} = 0 \rightarrow (4)$$

i) Wave equation interms of $\vec{E}$

Differentiating eq (2) with respect to time

$$\frac{\partial}{\partial t}(\nabla \times \vec{H}) = \frac{\partial^2 \vec{D}}{\partial t^2} \rightarrow (5)$$

changing order of integration at the L.H.S

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\partial^2 \vec{D}}{\partial t^2} \rightarrow (6)$$

But $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \nabla \times \frac{\partial \vec{H}}{\partial t} = \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow (7)$$

Taking curl of eq (1),

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t} \rightarrow (8)$$

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But $\vec{B} = \mu_0 \vec{H}$

$$\therefore \nabla \times \nabla \times \vec{E} = -\mu_0 \left(\nabla \times \frac{\partial \vec{H}}{\partial t} \right) \rightarrow (9)$$

Putting eq (7) in eq (9),

$$\nabla \times \nabla \times \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow (10)$$

Using vector identity,

$$\nabla \times \nabla \times \vec{E} = (\nabla . \vec{E}) \nabla - \nabla^2 \vec{E}$$

Hence eq (10) becomes

$$(\nabla . \vec{E}) \nabla - \nabla^2 \vec{E} = -\mu \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow (11)$$

From eq (3), $\nabla . \vec{D} = 0$

$$\therefore \nabla . \varepsilon_0 \vec{E} = 0$$

$$\therefore \nabla . \vec{E} = 0$$

Putting this value in eq (11),

$$\therefore -\nabla^2 \vec{E} = -\mu \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \rightarrow (12)$$

Eq (12) is called wave equation interms of electric field intensity $\vec{E}$.

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ii) Wave equation interms of $\vec{H}$:

Differentiating eq (1) w.r.t. time

$$\frac{\partial}{\partial t}(\nabla \times \vec{E}) = \frac{\partial}{\partial t}\left(-\frac{\partial \vec{B}}{\partial t}\right)$$

$$\frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\frac{\partial^2 \vec{B}}{\partial t^2} \rightarrow (13)$$

But $\vec{B} = \mu_0 \vec{H}$

$$\therefore \frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \rightarrow (14)$$

Changing the order of integration

$$\nabla \times \frac{\partial \vec{E}}{\partial t} = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \rightarrow (15)$$

Taking curl of eq (2),

$$\nabla \times \nabla \times \vec{H} = \nabla \times \frac{\partial \vec{D}}{\partial t} \rightarrow$$

But $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \nabla \times \nabla \times \vec{H} = \varepsilon_0 \nabla \times \frac{\partial \vec{E}}{\partial t} \rightarrow (16)$$

Using vector identity.

$$\nabla \times \nabla \times \vec{H} = (\nabla . \vec{H})\nabla - \nabla^2 \vec{H}$$

Eq (16) becomes

$$\therefore (\nabla . \vec{H})\nabla - \nabla^2 \vec{H} = \varepsilon_0 \nabla \times \frac{\partial \vec{E}}{\partial t} \rightarrow (17)$$

Putting eq (15) in to (17)

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$$(\nabla.\vec{H})\nabla - \nabla^2 \vec{H} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \rightarrow (18)$$

From eq (4)

$$\nabla.\vec{B} = 0$$

But $\vec{B} = \mu_0 \vec{H}$

$$\mu_0(\nabla.\vec{H}) = 0$$

$$\therefore \nabla.\vec{H} = 0$$

hence eq (18) becomes

$$-\nabla^2 \vec{H} = -\mu_0 \varepsilon_0 \frac{\nabla^2 \vec{H}}{\partial t^2}$$

$$\therefore \boxed{\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\nabla^2 \vec{H}}{\partial t^2}} \rightarrow (19)$$

iii) Eq (19) is called wave equation interms of magnetic field intensity $\vec{H}$.

iii) <u>Wave equation</u> <u>interms of $\vec{D}$</u>:

w.k.t $\vec{D} = \varepsilon_0 \vec{E}$

$$\therefore \vec{E} = \frac{\vec{D}}{\varepsilon_0}$$

Putting this value in eq (12),

$$\frac{\nabla^2 \vec{D}}{\varepsilon_0} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}(\vec{D}/\varepsilon_0)$$

$$\boxed{\nabla^2 \vec{D} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{D}}{\partial t^2}} \rightarrow (20)$$

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iv) Wave equation interms of $\vec{B}$

w.k.t, $\vec{B} = \mu_0 \vec{H}$

$$\therefore \vec{H} = \frac{\vec{B}}{\mu_0}$$

putting this value in eq (19)

$$\frac{\nabla^2 \vec{B}}{\mu_0} = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\left(\frac{\vec{B}}{\mu_0}\right)$$

$$\therefore \boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}} \longrightarrow (21)$$

* Standard form of wave equation:

$$\boxed{\nabla^2 A = \frac{1}{v^2} \frac{\partial^2 A}{\partial t^2}}$$

where $\frac{1}{v^2} = \mu_0 \varepsilon_0$

$$\therefore V = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \quad ; \quad \varepsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$$

$$\mu_0 = 4\pi \times 10^{-17} \text{ H/m}$$

$$\therefore v = \frac{1}{\sqrt{8.854 \times 10^{-12} \times 4\pi \times 10^{-17}}}$$

$$\boxed{c \text{ (or) } V = 3 \times 10^8 \text{ m/s}}$$

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Wave equations for conducting medium:

The maxwell's equations for the conducting medium are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow (1)$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

$$\nabla \cdot \vec{D} = 0 \rightarrow (3)$$

$$\nabla \cdot \vec{B} = 0 \rightarrow (4)$$

i) Wave equation interms of $\vec{E}$:-

Differentiating eq (2) w.r.t. time

$$\frac{\partial}{\partial t}(\nabla \times \vec{H}) = \frac{\partial \vec{J}}{\partial t} + \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \frac{\partial \vec{J}}{\partial t} + \frac{\partial^2 \vec{D}}{\partial t^2} \rightarrow (5)$$

But $\vec{D} = \varepsilon \vec{E}$ and $\vec{J} = \sigma \vec{E}$

Putting these values in eq (5)

$$\nabla \times \frac{\partial \vec{H}}{\partial t} = \sigma \frac{\partial \vec{E}}{\partial t} + \varepsilon \frac{\partial^2 \vec{D}}{\partial t^2} \rightarrow (6)$$

Taking curl of eq (1)

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{B}}{\partial t}$$

But $\vec{B} = \mu \vec{H}$

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$$\therefore \nabla \times \nabla \times \vec{E} = -\mu \; \nabla \times \frac{\partial \vec{H}}{\partial t} \rightarrow (7)$$

Using vector identity

$$\nabla \times \nabla \times \vec{E} = (\nabla \cdot \vec{E})\nabla - \nabla^2 \vec{E}$$

But $\vec{D} = \varepsilon \vec{E}$; $\vec{E} = \frac{\vec{D}}{\varepsilon}$

$$\therefore \nabla \times \nabla \times \vec{E} = \left(\nabla \cdot \frac{\vec{D}}{\varepsilon}\right) - \nabla^2 \vec{E}$$

from eq (3) , $\nabla \cdot \vec{D} = 0$

$$\therefore \nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} \rightarrow (8)$$

substitute eq (8) in (7)

$$-\nabla^2 \vec{E} = -\mu \left(\nabla \times \frac{\partial \vec{H}}{\partial t}\right) \rightarrow (9)$$

substitute eq (6) in to (9)

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0} \rightarrow (10)$$

Eq (10) is called as wave equation for good conductors interms of $\vec{E}$.

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ii) Wave equation interms of $\vec{H}$:

Differentiating eq 1 w.r.t. time

$$\frac{\partial}{\partial t}(\nabla \times \vec{E}) = -\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\nabla \times \frac{\partial \vec{E}}{\partial t} = -\frac{\partial^2 B}{\partial t^2}$$

But $B = \mu \vec{H}$

$$\therefore \nabla \times \frac{\partial \vec{E}}{\partial t} = -\mu \frac{\partial^2 \vec{H}}{\partial t^2} \rightarrow 11$$

Taking curl of eq 2,

$$\nabla \times \nabla \times \vec{H} = \nabla \times \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \rightarrow 12$$

Using vector identity,

$$\nabla \times \nabla \times \vec{H} = (\nabla \cdot \vec{H})\nabla - \nabla^2 \vec{H}$$

But $B = \mu \vec{H}$ $\therefore$ $\vec{H} = \frac{\vec{B}}{\mu}$

$$\nabla \times \nabla \times \vec{H} = \frac{1}{\mu}(\nabla \cdot \vec{B})\nabla - \nabla^2 \vec{H} \rightarrow 13$$

From eq 4, $\nabla \cdot \vec{B} = 0$

$$\therefore \nabla \times \nabla \times \vec{H} = -\nabla^2 \vec{H} \rightarrow 14$$

Substitute eq 14 in to 12

$$-\nabla^2 \vec{H} = \nabla \times \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

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But $\vec{J} = \sigma \vec{E}$ and $\vec{D} = \varepsilon \vec{E}$

$$\therefore -\nabla^2 \vec{H} = \nabla \times \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= \sigma(\nabla \times \vec{E}) + \varepsilon \left(\nabla \times \frac{\partial \vec{E}}{\partial t}\right) \rightarrow 15$$

Putting eq's 1 & 11 in 15

$$-\nabla^2 \vec{H} = \varepsilon\left(-\mu \frac{\partial^2 H}{\partial t^2}\right) + \sigma\left(\frac{-\partial \vec{B}}{\partial t}\right)$$

But $\vec{B} = \mu \vec{H}$

$$\therefore \nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{dt^2}$$

$$\boxed{\nabla^2 \vec{H} - \mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \rightarrow 16$$

Eq 16 is called wave equation interms of $\vec{H}$

iii) Wave equation interms of $\vec{D}$:-

w.k.t $\vec{B} = \varepsilon \vec{E}$

$$\therefore \vec{E} = \frac{\vec{D}}{\varepsilon}$$

Putting this value in eq 10,

$$\nabla^2 \frac{\vec{D}}{\varepsilon} - \frac{\mu\sigma}{\varepsilon} \frac{\partial \vec{D}}{\partial t} - \frac{\mu\varepsilon}{\varepsilon} \frac{\partial^2 \vec{D}}{\partial t^2} = 0$$

$$\therefore \boxed{\nabla^2 \vec{D} - \mu\sigma \frac{\partial \vec{D}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{D}}{\partial t^2} = 0} \rightarrow 17$$

Eq 17 is called wave equation for good conductors interms of $\vec{D}$.

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iv) wave equation interms of $\vec{B}$:

w.k.t. $\vec{B} = \mu \vec{H}$

$\vec{H} = \vec{B}/\mu$

Putting this value in eq 16,

$$\frac{\nabla^2 \vec{B}}{\mu} - \frac{\mu\sigma}{\mu}\frac{\partial \vec{B}}{\partial t} - \frac{\mu\varepsilon}{\mu}\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{B} - \mu\sigma \frac{\partial \vec{B}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}} \rightarrow (18)$$

Eq 18 is called wave equation for good conductors interm of $\vec{B}$.

---

* Wave equations in phasor form :

Recall wave equation for free space.

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \rightarrow (1)$$

For a sinusoidal time variation the wave equation is obtained by replacing $\frac{\partial}{\partial t}$ by the term $j\omega$.

$$\therefore \nabla^2 \vec{E} = \mu_0 \varepsilon_0 (j\omega)^2 \vec{E}$$

$$= \mu_0 \varepsilon_0 j^2 \omega^2 \vec{E} \qquad \therefore j^2 = -1$$

$$\therefore \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \omega^2 \vec{E} \rightarrow (2)$$

Put $\beta^2 = \omega^2 \mu_0 \varepsilon_0$

$$\therefore \nabla^2 \vec{E} = -\beta^2 \vec{E} \rightarrow (3)$$

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Eq $\nabla^2 \vec{E} + \beta^2 \vec{E} = 0 \rightarrow (4)$

Eq (4) is called as vector Helmholtz equation.

Expanding eq (3)

$$\frac{\partial^2 E_x}{\partial z^2}\vec{a_x} + \frac{\partial^2 E_y}{\partial z^2}\vec{a_y} + \frac{\partial^2 E_z}{\partial z^2}\vec{a_z}$$

$$= -\beta^2 (E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z}) \rightarrow (5)$$

Assume that wave is travelling in z-direction,

$\therefore \quad E_z = 0$

~~Hence eq (5) becomes~~,

$$\frac{\partial^2 E_x}{\partial z^2}\vec{a_x} + \frac{\partial^2 E_y}{\partial z^2}\vec{a_y} = -\beta^2 E_x \vec{a_x} - \beta^2 E_y \vec{a_y} \rightarrow (6)$$

Equating the coefficients of eq (6)

$$\frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_x \rightarrow (7)$$

$$\frac{\partial^2 E_y}{\partial z^2} = -\beta^2 E_y \rightarrow (8)$$

Similarly for the magnetic field intensity $(\bar{H})$, we can write,

$$\frac{\partial^2 H_x}{\partial z^2} = -\beta^2 H_x \rightarrow (9)$$

$$\frac{\partial^2 H_y}{\partial z^2} = -\beta^2 H_y \rightarrow (10)$$

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Now consider E_y component only. Then the solution
of eq 8 can be written as

$$E_y = c_1 e^{-j\beta z} + c_2 e^{j\beta z} \rightarrow 11$$

where
$c_1 = c_2 =$ constants

Eq 11 gives the solution in phasor quantity.

In terms of time varying form, the E_y component of the wave can be written as

$$\tilde{E}_y(z,t) = Re\{c_1 e^{-\beta z} + c_2 e^{j\beta z}).e^{j\omega t}$$

$$= Re\{c_1 e^{-j\beta z} e^{j\omega t} + c_2 e^{j\beta z} e^{j\omega t}\}$$

$$= Re\{c_1 e^{j(\omega t - \beta z)} + c_2 e^{j(\omega t + \beta z)}\} \rightarrow 12$$

By using trignometric identities

$$e^{+j(\omega t - \beta z)} = \cos(\omega t - \beta z) + j \sin(\omega t - \beta z) \rightarrow 13$$

$$e^{j(\omega t + \beta z)} = \cos(\omega t + \beta z) + j \sin(\omega t + \beta z) \rightarrow 14$$

Substitute eq's 13 & 14 in 12

$$\tilde{E}(z,t) = Re\{c_1[\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)] + c_2[\cos(\omega t + \beta z) + j \sin(\omega t + \beta z)]\}$$

$$\boxed{\tilde{E}(z,t) = c_1 \cos(\omega t - \beta z) + c_2 \cos(\omega t + \beta z)} \rightarrow 15$$

Eq 15 gives the solution of wave equation interms of time varying fields & indicates sum of two waves

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travelling in the opposite direction

Let $\omega t - \beta z = A$

$$\omega - \beta \frac{dz}{dt} = 0$$

$$\therefore \omega = \beta \frac{dz}{dt} \quad ; \quad \frac{dz}{dt} = v$$

$$\therefore v = \frac{\omega}{\beta}$$

From eq ③, $\beta^2 = \omega^2 \mu_0 \varepsilon_0$

$$\therefore \beta = \omega \sqrt{\mu_0 \varepsilon_0}$$

$$\boxed{v = \frac{\omega}{\omega \sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C = 3 \times 10^8 \ m/s}$$

wave impedance (2) (or) intrinsic impedance :

$$\eta = \frac{E}{H} = \sqrt{\frac{\mu}{\varepsilon}}$$

For free space

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \quad ; \quad \mu_r = 1 \ \& \ \varepsilon_r = 1$$

$$= \sqrt{\frac{4\pi \times 10^{-7}}{8.854 \times 10^{-12}}}$$

$$\therefore \boxed{\eta = 377 \ \Omega = 120\pi}$$

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* Potential function (or) Retarded potential :-

Time varying potentials are also called as retarded potentials (or) potential functions.

For static electric fields, electric scalar potential is given by

$$V = \int_V \frac{\rho_v}{4\pi\varepsilon R} dv \rightarrow (1)$$

For static magnetic field, the magnetic vector potential is given by

$$\vec{A} = \int_V \frac{\mu \vec{J}}{4\pi R} dv \rightarrow (2)$$

For time varying fields,

$$\left.\begin{aligned} \vec{E} &= -\nabla V \\ \vec{B} &= \nabla \times \vec{A} \end{aligned}\right\} \rightarrow (3)$$

If we combine above relation with the expression of Faraday's law

$$\vec{E} = -\nabla V + N \rightarrow (4)$$

where

N - unknown value

$$\nabla \times N = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \text{point form of Faraday's law}$$

$$\nabla \times N = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$$

$$= -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

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$$\therefore N = -\frac{\partial \vec{A}}{\partial t}$$

$$\therefore \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \rightarrow (5)$$

and also

$$\vec{D} = \varepsilon \vec{E}$$

$$\nabla \cdot \vec{D} = \rho_v$$

$$\varepsilon \nabla \cdot \vec{E} = \rho_v$$

$$\therefore \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \rightarrow (6)$$

By taking divergence of eq (5) & using eq (6).

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A})$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho_v}{\varepsilon} \rightarrow (7)$$

Consider maxwell's Equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

But $B = \mu \vec{H}$ and $\vec{D} = \varepsilon \vec{E}$

Taking curl of eq (5),

$$\nabla \times \nabla \times \vec{A} = \nabla \times \vec{B} = \mu \nabla \times \vec{H}$$

$$(\nabla \times \nabla \times \vec{A}) = \mu \vec{J} + \mu \varepsilon \frac{\partial \vec{E}}{\partial t} \rightarrow (8)$$

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From eq (5),

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + \mu \varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \mu J - \mu \varepsilon \nabla \frac{\partial V}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} \rightarrow (9)$$

Using vector identity

$$\nabla \times \nabla \times \vec{A} = (\nabla . \vec{A})\nabla - \nabla^2 \vec{A}$$

$$\therefore (\nabla . \vec{A})\nabla - \nabla^2 \vec{A} = \mu \vec{J} - \mu \varepsilon \nabla \frac{\partial V}{\partial t} - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \nabla(\nabla . \vec{A}) = -\mu \vec{J} + \mu \varepsilon \nabla \frac{\partial V}{\partial t} + \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} \rightarrow (10)$$

From eq (10),

$$\nabla . \vec{A} = -\mu \varepsilon \frac{\partial V}{\partial t} \rightarrow (11)$$

Eq (11) is called Lorentz Condition for potentials.

Using eq (11) in (7)

$$\nabla^2 V + \frac{\partial}{\partial t}\left(-\mu \varepsilon \frac{\partial V}{\partial t}\right) = -\frac{\rho_v}{\varepsilon}$$

$$\nabla^2 V - \mu \varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\varepsilon} \rightarrow (12)$$

Using Lorentz condition in eq (10),

$$\nabla^2 \vec{A} - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \rightarrow (13)$$

Eq's (12) & (13) are called wave equations.

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The solution to the eqn's 12 & 13 are

$$V = \int_v \frac{[\rho_v]}{4\pi\varepsilon R} dv \rightarrow (14)$$

$$\vec{A} = \int \frac{\mu[\vec{J}]}{4\pi R} dv \rightarrow (15)$$

The $[\rho_v]$ and $[\vec{J}]$ mean that the time is $\rho_v(x, y, z, t)$ (or) $J(x, y, z, t)$ is replaced by retarded time t'.

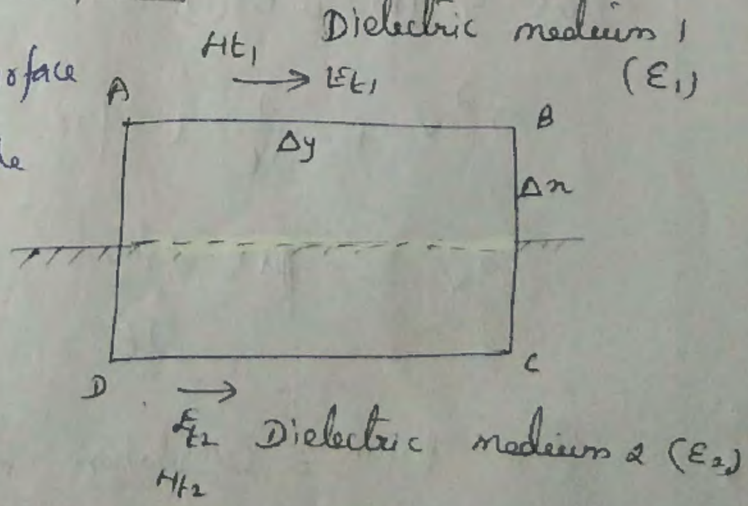
$$t' = t - \frac{R}{v} \rightarrow (16)$$

; $R = \bar{r} - \bar{r}'$ = distance between the differential element of charge & point P.

* Electromagnetic Boundary conditions:

To find tangential components.

Consider a closed surface (rectangle path) at the boundary between two dielectric media.



W.k.t

$$V = \oint \vec{E} \cdot d\vec{l} = 0 \rightarrow (1)$$

For path A-B-C-D,

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$$\oint \vec{E}\cdot d\vec{l} = \int_A^B \vec{E}\cdot d\vec{l} + \int_B^C \vec{E}\cdot d\vec{l} + \int_C^D \vec{E}\cdot d\vec{l} + \int_D^A \vec{E}\cdot d\vec{l} = 0 \quad \rightarrow (2)$$

For paths B-C and D-A, length Δx approaches zero.

$$\therefore E_{t1}\,\Delta y - E_{t2}\,\Delta y = 0$$

$$(E_{t1} - E_{t2})\,\Delta y = 0$$

$$\therefore \boxed{E_{t1} = E_{t2}}$$

The integral form of Maxwell's Equation I is

$$\oint_l \vec{H}\cdot d\vec{l} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\cdot d\vec{s}$$

For path A-B-C-D,

$$\oint_l \vec{H}\cdot d\vec{l} = \int_A^B \vec{H}\cdot d\vec{l} + \int_B^C \vec{H}\cdot d\vec{l} + \int_C^D \vec{H}\cdot d\vec{l} + \int_D^A \vec{H}\cdot d\vec{l}$$

$$= \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)\Delta x\,\Delta y$$

For path B-C and D-A, $\Delta x \to 0$

$$\vec{H}_{t1}\,\Delta y - \vec{H}_{t2}\,\Delta y = \vec{J}\,\Delta x\,\Delta y + \frac{\partial \vec{D}}{\partial t}\,\Delta x\,\Delta y$$

But $\lim_{\Delta x \to 0} \vec{J}\,\Delta x = J_l$

$$(H_{t1} - H_{t2})\,\Delta y = J_l\,\Delta y$$

$$\boxed{H_{t1} - H_{t2} = J_l}$$

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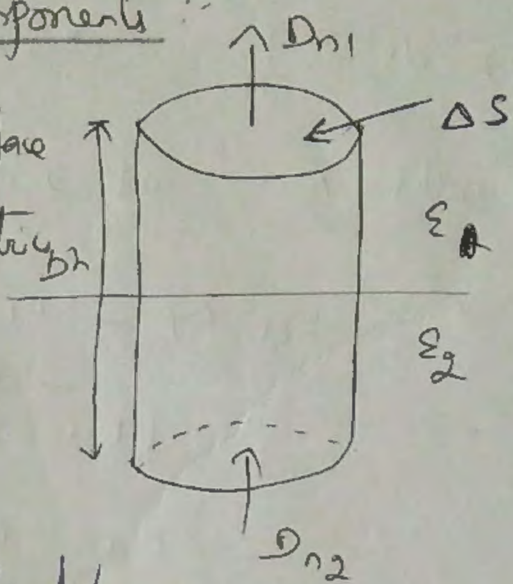
To find normal components :-



Consider a Gaussian surface at boundary of two dielectrics

The integral form of Maxwell's equation III is

$$\int_S \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv$$

$$\int_{Top} \vec{D} \cdot \vec{ds} + \int_{Bottom} \vec{D} \cdot \vec{ds} = \int_V \rho_v \, dv$$

$$D_{n1} \Delta S - D_{n2} \Delta S = Q$$

$$(D_{n1} - D_{n2}) = \frac{Q}{\Delta S} \qquad \therefore \rho_s = \frac{Q}{\Delta S}$$

$$\boxed{D_{n1} - D_{n2} = \rho_s}$$

The integral form of Maxwell's equation IV is

$$\int_S \vec{B} \cdot \vec{ds} = 0$$

$$\int_{Top} \vec{B} \cdot \vec{ds} + \int_{Bottom} \vec{B} \cdot \vec{ds} = 0$$

$$B_{n1} \Delta S - B_{n2} \Delta S = 0$$

$$\boxed{B_{n1} = B_{n2}}$$

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Time Harmonic Fields:

A time harmonic field is varies periodically or sinusoidally with time. Sinusoids are easily expressed in phasors.

A phasor is a complex number, a phasor z can be represented as

$$z = x + jy = r \angle \phi$$

(or)

$$z = r e^{j\phi} = r(\cos\phi + j\sin\phi)$$

where $j = \sqrt{-1}$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y/x)$$

The phasor can be represented in rectangular form or in polar form.

$$z = x + jy = r \angle \phi$$

$$z_1 = x_1 + jy_1 = r_1 \angle \phi_1 \quad ; \quad z_2 = x_2 + jy_2 = r_2 \angle \phi_2$$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} \angle \phi_1 - \phi_2$

Square root: $\sqrt{z} = \sqrt{r} \angle \phi/2$

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Complex conjugate $z^* = x - jy = r\angle -\phi = re^{-j\phi}$

For an time element, $\phi = \omega t + \theta$

$$Re(re^{j\phi}) = r(\cos(\omega t + \theta)$$

$$Im(re^{j\phi}) = r\sin(\omega t + \theta)$$

Time harmonic Maxwell's equations:

Point form:	Integral form
$\nabla \cdot \vec{D} = \rho_v$	$\oint \vec{D} \cdot d\vec{s} = \int \rho_v \, dv$
$\nabla \cdot B = 0$	$\oint \vec{B} \cdot d\vec{s} = 0$
$\nabla \times \vec{E} = -j\omega B$	$\oint \vec{E} \cdot d\vec{l} = -j\omega \int \vec{B} \cdot d\vec{s}$
$\nabla \times \vec{H} = \vec{J} + j\omega\vec{D}$	$\oint \vec{H} \cdot d\vec{l} = \int (\vec{J} + j\omega\vec{D}) \cdot d\vec{s}$

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EC8451 - Electromagnetic Fields

UNIT - V

Plane Electromagnetic waves

Plane waves in lossless media, plane waves in lossy media (low-loss dielectric and good conductors), Group velocity, Electromagnetic power flow and poynting vector, Normal incidence at a plane conducting boundary, Normal incidence at a plane dielectric boundary.

Waves:

Waves are means of transporting energy or information.

Plane waves in Lossy dielectrics:

In practical situations, dielectric material do posses certain amount of conductivity, although in some cases it may be negligible in effect.

Maxwell's equation from Ampere's law may be written as

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$= \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \rightarrow (1)$$

$\because \vec{J} = \sigma \vec{E}$

$\vec{D} = \varepsilon \vec{E}$

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Put $\frac{\partial}{\partial t} = j\omega$

$$\nabla \times \vec{H} = \sigma \vec{E} + j\omega\varepsilon \vec{E}$$

$$\nabla \times \vec{H} = (\sigma + j\omega\varepsilon)\vec{E} \rightarrow (2)$$

Maxwell's equation from Faraday's law may be written as

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \because \vec{B} = \mu \vec{H}$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Put $\partial/\partial t = j\omega$

$$\therefore \nabla \times \vec{E} = -j\omega\mu \vec{H} \rightarrow (3)$$

Take curl on both sides of eq (3)

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu \; \nabla \times \vec{H} \rightarrow (4)$$

Substitute eq (2) in to (4)

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu\,(\sigma + j\omega\varepsilon)\vec{E}$$

Using vector identity

$$(\nabla . \vec{E})\nabla - \nabla^2 \vec{E} = -j\omega\mu\,(\sigma + j\omega\varepsilon)\vec{E}$$

But $\nabla . \vec{E} = 0$

$$\therefore \cancel{-}\nabla^2 \vec{E} = \cancel{-}j\omega\mu\,(\sigma + j\omega\varepsilon)\vec{E} \rightarrow (5)$$

But $\gamma^2 = j\omega\mu\,(\sigma + j\omega\varepsilon)$

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$$\nabla^2 \vec{E} = \gamma^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} - \gamma^2 \vec{E} = 0 \rightarrow (6)$$

where γ = Propagation constant

Similarly for magnetic field

$$\nabla^2 \vec{H} - \gamma^2 \vec{H} = 0 \rightarrow (7)$$

Propagation constant may be expressed as

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= \sqrt{j\omega\mu\sigma + (j\omega)^2 \mu\varepsilon}$$

$$= \sqrt{\frac{(j\omega)^2}{j\omega}\mu\sigma + (j\omega)^2 \mu\varepsilon}$$

$$= j\omega\sqrt{\mu\varepsilon - j\frac{\mu\sigma}{\omega}}$$

$$= j\omega\sqrt{\mu\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]}$$

The term $\frac{\sigma}{\omega\varepsilon}$ is a 'loss tangent'

$$\therefore \gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{1 - \frac{j\sigma}{\omega\varepsilon}} = \alpha + j\beta \rightarrow (8)$$

Expanding the second radical by applying Binomial theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots, \quad |x| < 1$$

Let $x = \left(\frac{-j\sigma}{\omega\varepsilon}\right)$; $n = 1/2$

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$$\therefore \gamma = j\omega\sqrt{\mu\varepsilon}\left[1 + \frac{\sigma j}{2\omega\varepsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + \cdots\right]$$

$$= j\omega\sqrt{\mu\varepsilon} + \frac{\omega\sqrt{\mu\varepsilon}\,\sigma}{2\omega\varepsilon} + j\omega\sqrt{\mu\varepsilon}\,\frac{1}{8}\left(\frac{\sigma^2}{\omega^2\varepsilon^2}\right) + \cdots$$

$$= j\omega\sqrt{\mu\varepsilon} + \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}\,\frac{1}{8}\left(\frac{\sigma^2}{\omega^2\varepsilon^2}\right) + \cdots$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\,\frac{\sigma^2}{\omega^2\varepsilon^2} + \cdots\right]$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}$$ $\therefore$ (neglecting higher order terms)

$$\gamma = \alpha + j\beta$$

Comparing real and imaginary parts

Attenuation constant, $\alpha = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$

Phase constant, $\beta = \omega\sqrt{\mu\varepsilon}$

$\therefore$ Intrinsic constan, $\eta = \frac{E}{H}$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

$$= \sqrt{\frac{j\omega\mu}{j\omega\varepsilon\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)}} = \sqrt{\frac{\mu}{\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{-1}}$$

$$= \sqrt{\frac{\mu}{\varepsilon}\left(1 - \frac{\sigma}{j\omega\varepsilon}\right)} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 + \frac{j\sigma}{\omega\varepsilon}\right)^{1/2}$$

$$= \sqrt{\frac{\mu}{\varepsilon}}\left(1 + \frac{j\sigma}{2\omega\varepsilon}\right)$$

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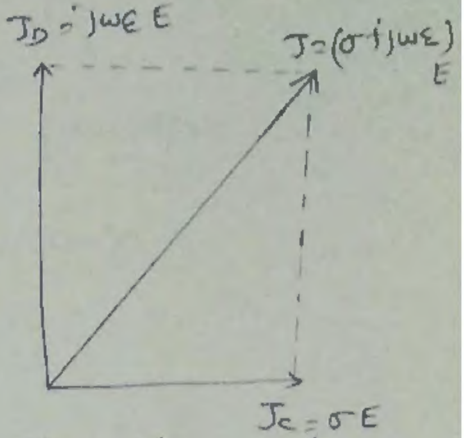
* Loss tangent :

$$\left|\frac{J_c}{J_D}\right| = \left|\frac{\sigma}{j\omega\varepsilon}\right| = \frac{\sigma}{\omega\varepsilon}$$

$$\therefore \tan\theta = \frac{\sigma}{\omega\varepsilon}$$



Time phase relation between J_c, J_D & J

Fig. shows that the displacement current density leads conduction current density by 90°

Loss tangent is a measure of impurity of the given dielectric material or medium. The loss tangent is generally constant irrespective of frequency.

Waves in lossless dielectric or perfect dielectric :

Consider that the uniform plane wave is propagating through a perfect dielectric. If the medium is perfect dielectric, then its properties are given by

$$\sigma = 0 \; ; \; \mu = \mu_0 \mu_r \; ; \; \varepsilon = \varepsilon_0 \varepsilon_r$$

The propagation constant is given by

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

For lossless dielectric, $\sigma = 0$

$$\gamma = \sqrt{j\omega\mu(j\omega\varepsilon)} = j\omega\sqrt{\mu\varepsilon}$$

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$$\therefore \gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon}$$

Comparing real and imaginary parts

$$\alpha = 0 \quad ; \quad \beta = \omega\sqrt{\mu\varepsilon}$$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{0 + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}}$$

$$= \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}} = \sqrt{\frac{\mu}{\varepsilon}}\angle 0^\circ$$

(E and H are in time phase with each other)

For free space, $\varepsilon_r = 1$; $\mu_r = 1$

$$\therefore \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 3\pi\ \Omega \quad (\text{or}) \quad 120\pi$$

and $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$$

Waves in good conductors :-

A practical or good conductors are material which has very high conductivity.

For good conductors

$$\frac{\sigma}{\omega\varepsilon} >> 1 \quad ; \quad \sigma = \infty, \ \varepsilon = \varepsilon_0, \ \mu = \mu_0\mu_r$$

The propagation constant γ is given by

$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

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As $\sigma >> \omega\varepsilon$, we can neglect imaginary part $(j\omega\varepsilon)$

$$\gamma = \sqrt{j\omega\mu\sigma} = \sqrt{\omega\mu\sigma}\,\sqrt{j}$$

But $j = 1\angle 90^\circ$

$$\gamma = \sqrt{\omega\mu\sigma}\,\sqrt{1\angle 90^\circ} = \sqrt{\omega\mu\sigma}\;\angle 45^\circ$$

$$= \sqrt{\omega\mu\sigma}\,[\cos 45^\circ + j\sin 45^\circ]$$

$$= \sqrt{\omega\mu\sigma}\,[1/\sqrt{2} + j\,1/\sqrt{2}]$$

$$= \sqrt{\omega\mu\sigma}\,[\tfrac{1}{\sqrt{2}}(1+j1)]$$

$$= \sqrt{2\pi f\mu\sigma}\,[1/\sqrt{2}\,(1+j1)]$$

$$\gamma = \alpha + j\beta = \sqrt{\pi f\mu\sigma} + j\sqrt{\pi f\mu\sigma}$$

Thus for good conductor

$$\alpha = \beta =$$

$$\therefore \alpha = \sqrt{\pi f\mu\sigma}\ \ Np/m$$

$$\beta = \sqrt{\pi f\mu\sigma}\ \ rad/m$$

The intrinsic impedance of a good conductor is given by

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

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For good conductor $\sigma >> j\omega\varepsilon$

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\sqrt{j}$$

But $\sqrt{j} = \sqrt{1\angle 90^\circ} = 1\angle 45^\circ = \cos 45^\circ + j \sin 45^\circ$

$$\sqrt{j} = 1/\sqrt{2} + j\, 1/\sqrt{2}$$

$$\therefore \eta = \sqrt{\frac{\omega\mu}{\sigma}}\left[1/\sqrt{2} + j\, 1/\sqrt{2}\right] = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$$

$$= \sqrt{\frac{\omega\mu}{2\sigma}}(1+j1) = \sqrt{\frac{2\pi f\mu}{2\sigma}}(1+j1)$$

$$\eta = \sqrt{\frac{\pi f\mu}{\sigma}}(1+j1)$$

$\therefore$ The angle of intrinsic impedance is 45°.

It is found that in good conductors, α and β are large since σ is large.

The wave is attenuated greatly as it progresses through the conductor. But velocity and characteristic impedance are reduced considerably.

Skin depth :

At radio frequencies, the rate of attenuation is very large and the wave may penetrate only a very short distance before being reduced to a negligible small value.

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High frequency signal is not propagated in the interior of the conductor. It travels only on the surface (skin) of the conductor. This is known as skin effect.

"The depth of penetration or skin depth (δ) is defined as the depth into which the wave has been attenuated to $\frac{1}{e}$ or approximately 37% of its original value.

By definition,

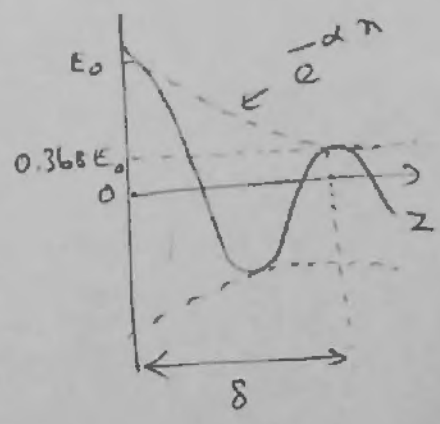
$$e^{-\alpha x} = \frac{1}{e}$$

But $x = \delta$

$$e^{-\alpha \delta} = 1/e$$

$$\alpha \delta = 1$$

$$\therefore \delta = 1/\alpha = \frac{1}{\sqrt{\pi f \mu \sigma}}$$



Skin depth

Since

$\alpha = \beta$ in good conductor

$$\delta = \frac{1}{\beta} = \frac{\lambda}{2\pi}$$

and also

$$\eta = \frac{1+j}{\sigma \delta}$$

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* **Poynting vector and Poynting Theorem :-**

Energy can be transported from one point to another point by means of EM waves. The rate of transportation can be obtained from Maxwell's equations.

Poynting's Theorem:

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in energy stored within v minus the ohmic losses.

$$\vec{P} = \vec{E} \times \vec{H} \quad W/m^2 \text{ (or) } VA/m^2 \rightarrow ①$$

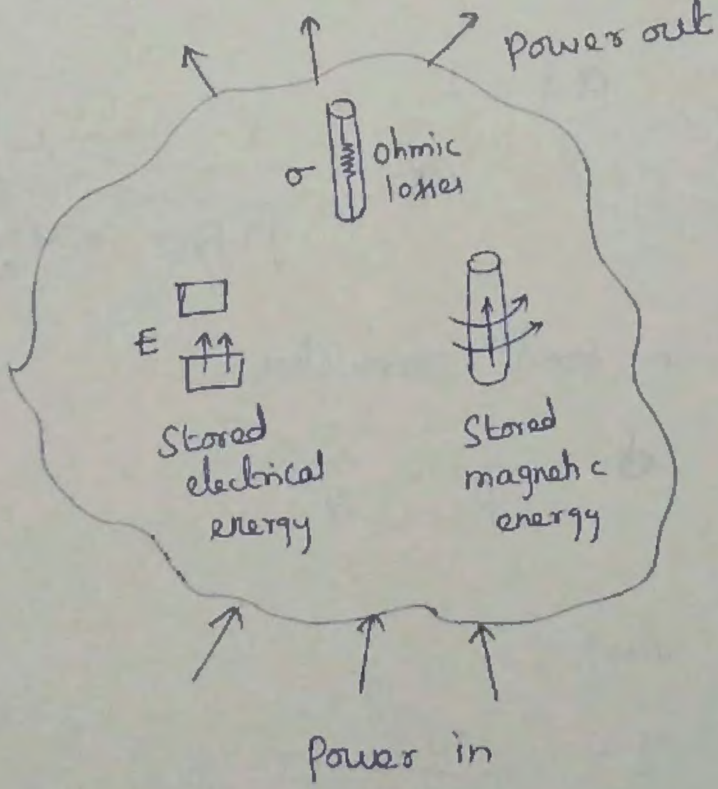

Illustration of Power balance for EM fields

Proof:

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According to Maxwell's 1st equation for time varying fields

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

$$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \rightarrow (2)$$

Taking dot product on both sides of eq (2), with $\vec{E}$,

$$\vec{E} \cdot \vec{J} = \vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \rightarrow (3)$$

By using vector identity, $\nabla \cdot (\vec{A} \times \vec{B}) = B \cdot (\nabla \times \vec{A}) - \vec{A} (\nabla \times \vec{B})$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) \rightarrow (4)$$

Substitute eq (4) in to eq (3),

$$\vec{E} \cdot \vec{J} = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \rightarrow (5)$$

But $\vec{D} = \varepsilon \vec{E}$

$$\vec{E} \cdot \vec{J} = \vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \rightarrow (6)$$

According to Maxwell's 2nd equation,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

But $\vec{B} = \mu \vec{H}$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \rightarrow (7)$$

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Substitute eq 7 into 6,

$$\vec{E}\cdot\vec{J} = -\mu\,\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} - \nabla\cdot(\vec{E}\times\vec{H}) - \varepsilon\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} \rightarrow (8)$$

Now, consider the term $\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H})$ can be expressed as

$$\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) = \vec{H}\cdot\frac{\partial\vec{H}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{H}}{\partial t}$$

$$= 2\vec{H}\cdot\frac{\partial\vec{H}}{\partial t}$$

$$\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}H^2 \rightarrow (9)$$

Similarly

$$\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t} \rightarrow (10)$$

Substitute eq's 9 and 10 into eq 8,

$$\vec{E}\cdot\vec{J} = -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \nabla\cdot(\vec{E}\times\vec{H}) - \frac{\varepsilon}{2}\frac{\partial E^2}{\partial t}$$

$$\vec{E}\cdot\vec{J} = -\frac{\partial}{\partial t}\left[\mu\frac{H^2}{2} + \varepsilon\frac{E^2}{2}\right] - \nabla\cdot(\vec{E}\times\vec{H}) \rightarrow (11)$$

Taking volume integral of eq 11

$$\int_v \vec{E}\cdot\vec{J}\,dv = -\frac{\partial}{\partial t}\int_v\left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2}\right)dv - \int_v \nabla\cdot(\vec{E}\times\vec{H})\,dv \rightarrow (12)$$

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Applying divergence theorem,

$$\int_V \nabla \cdot (\vec{E} \times \vec{H})\, dv = \oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s}$$

$$\therefore \int_V \vec{E} \cdot \vec{J}\, dv = -\frac{\partial}{\partial t} \int_V \left(\frac{\mu H^2}{2} + \frac{\varepsilon E^2}{2} \right) dv - \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s} \rightarrow (13)$$

$$\int_S \vec{E} \times \vec{H} \cdot ds = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv - \int_V \vec{E} \cdot \vec{J}\, dv$$

Put $\vec{J} = \sigma \vec{E}$

$$\therefore \int_S (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv - \int_V \sigma E^2\, dv \rightarrow (14)$$

Total Power leaving the volume = Rate of decrease in energy stored in electric and magnetic fields − Ohmic power dissipated

The quantity $E \times H$ on the L.H.S. of eq (14) is known as the poynting vector P, measured in watts per square meter (W/m^2)

$$\therefore \vec{P} = \vec{E} \times \vec{H} \rightarrow \text{Instantaneous power density vector.}$$

The value of average poynting vector is

$$\vec{P} = \frac{1}{2} Re \left(\vec{E} \times \vec{H}^* \right) \rightarrow (15)$$

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where H^* = complex conjugate of $\vec{H}$

In complex notation, instantaneous poynting vector is represented as

$$\bar{P}_{inst} = \vec{P}_{real} + j\,\vec{P}_{react}$$

where $\vec{P}_{real} = P_{avg} = \frac{1}{2}\,Re\,(\vec{E} \times \vec{H}^*)$

$$\vec{P}_{react} = \frac{1}{2}\,Im\,(\vec{E} \times \vec{H}^*)$$

Interms of voltage and current,

$$\vec{P}_{real} = \frac{|V|\,|I|}{2}\cos\theta$$

$$\vec{P}_{react} = \frac{|V|\,|I|}{2}\sin\theta$$

---

* Power flow in a coaxial cable:

Consider a coaxial cable of inner radius 'a' which carries a current I.

According to poynting theorem,

$$\vec{P} = \vec{E} \times \vec{H} \quad \rightarrow (1)$$

But $\vec{J} = \sigma\vec{E}$

$$\vec{E} = \frac{\vec{J}}{\sigma} \quad \rightarrow (2)$$

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Substitute eq ② into eq ①,

$$\vec{P} = \frac{\vec{J}}{\sigma} \times \vec{H} \rightarrow ③$$

Consider the current density is in Z-direct and $\vec{H}$ will be along $\vec{a}_\phi$ direction

$$\vec{J} = J\,\vec{a}_z \rightarrow ④$$

$$\vec{H} = \vec{H}\,\vec{a}_\phi \rightarrow ⑤$$

Hence eq ③ becomes

$$\vec{P} = \frac{J}{\sigma} H\,(\vec{a}_z \times \vec{a}_\phi) = \frac{-J}{\sigma} H\,\vec{a}_r \rightarrow ⑥$$

But $\vec{a}_z \times \vec{a}_\phi = -a_r$

In case of coaxial cable, magnetic field intensity at any distance 'r' is

$$H = \frac{I}{2\pi r} \rightarrow ⑦$$

But $I = J.A = J.\pi r^2$

$$\therefore H = \frac{J.\pi r^2}{2\pi r} = \frac{J.r}{2} \rightarrow ⑧$$

Substitute eq ⑧ into eq ⑥,

$$\vec{P} = \frac{-J}{\sigma}\left(\frac{J.r}{2}\right)\vec{a}_r$$

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$$\vec{P} = -\frac{J^2 r}{2\sigma} \vec{a_r} \rightarrow (9)$$

Now, total power flow is obtained by integrating eq (9) over a cylindrical surface of radius $(r = a)$.

For this surface the magnitude of Poynting vector is

$$P = \frac{J^2 a}{2\sigma} \rightarrow (10)$$

$\therefore$ Total power flow = $P \times$ area of cylindrical surface

$$= \frac{J^2 a}{2\sigma} \times 2\pi a L$$

$$= \left(\frac{JL}{\sigma}\right)\left(J \pi a^2\right) \rightarrow (11)$$

From eq (11),

First bracket term indicates

$E.L = V \Rightarrow$ voltage as $\frac{J}{\sigma}$

Second bracket term indicates

$J \times \text{Area} = I$

$\therefore$ Total power flow $= V.I \Rightarrow$ power dissipated in coaxial cable

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* Reflection by a perfect conductor :

Uniform plane waves travelling in unbounded and homogeneous media.

When uniform plane wave travels from one medium to other having different intrinsic impedance, the reflection takes place at the boundary.

The part of wave is transmitted in medium-2 and remaining part is reflected back to medium-1, depending on the conductive parameter, η, ε, μ etc of media.

Depending upon manner in which the uniform plane wave is incident on the boundary, there are two cases of incidence.

i) Normal incidence :

When a uniform plane wave incidence normally to the boundary between the media, then it is known as normal incidence.

ii) Oblique incidence :

When a uniform plane wave incidence obliquely to the boundary between the media, then it is known as oblique incidence.

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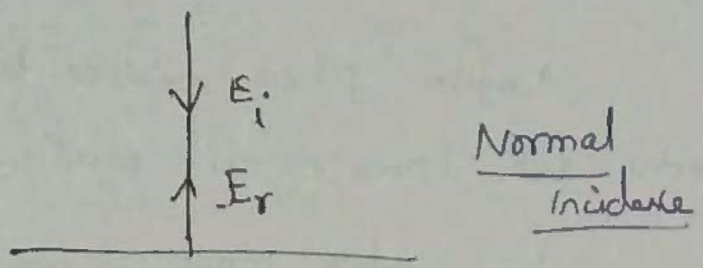
* Wave incident normally on a perfect conductor:-



When the plane wave incident normally upon the surface of perfect conductor, the wave is entirely reflected. Since there can be no loss with in a perfect conductor, none of the energy is absorbed. This is shown in figure.

As a result, the amplitude of E and H in the incident wave are the same as in reflected wave and differ by π (out of phase) i.e, $E_i = -E_r$.

Let the electric field of incident wave is $E_i e^{\gamma x}$ since attenuation is zero $(\alpha = 0)$, the propagation constant becomes $\gamma = j\beta$. Then the incident wave is $E_i e^{-j\beta x}$ and the reflected wave is $e^{j\beta x}$

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The resultant electric field is sum of electric field of incident and reflected waves.

$$E_T(x) = E_i e^{-j\beta x} + E_r e^{j\beta x} \rightarrow (1)$$

But $E_i = -E_r$

$$E_T(x) = E_i (e^{-j\beta x} - e^{j\beta x})$$

$$= -2j E_i \sin\beta x \rightarrow (2)$$

$\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Expressing eq (2) in time variation

$$E_T(x, t) = -2j E_i \sin\beta x \, e^{j\omega t} \rightarrow (3)$$

If E_i is chosen to be real

$$\therefore E_T(x,t) = 2 E_i \sin\beta x \sin\omega t \rightarrow (4)$$

Eq (4) shows that the incident & reflected waves combine to produce a standing waves, which does not progress.

Similarly,

$$H_T(x) = H_i e^{-j\beta x} + H_r e^{j\beta x}$$

But $H_i = H_r$

$$H_T(x) = H_i (e^{-j\beta x} + e^{j\beta x})$$

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$$H_T(x) = 2H_i \cos\beta x \rightarrow (5)$$

Expressing Eq (5) in time variation

$$H_T(x,t) = 2H_i \cos\beta x \, e^{j\omega t} \rightarrow (6)$$

if H_i is chosen to be real

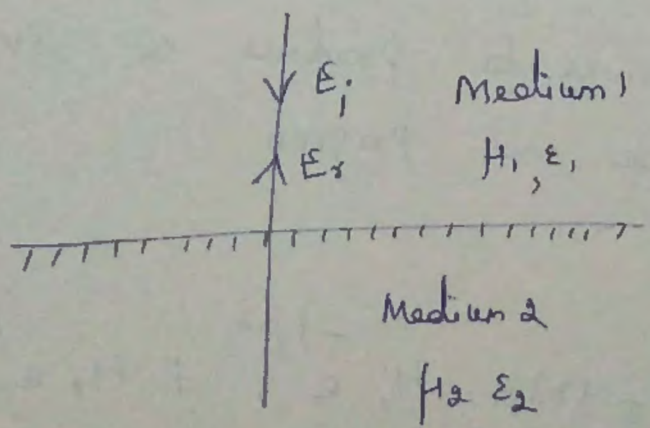
$$H_T(x,t) = 2H_i \cos\beta x \cos\omega t \rightarrow (7)$$

Eq (7) shows that the magnetic field H has a standing wave distribution.

It indicates that E and H are differ $\frac{\pi}{2}$ in phase.

---

* Wave incident normally on a perfect dielectric :-



Normal incidence

Consider two perfect dielectric media, Separated by a boundary as shown in Fig.

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Let ε_1, μ_1 are permittivity and permeability of medium 1 and ε_2, μ_2 are permittivity and permeability of medium 2

Let E_i – Electric field of incident wave
E_r – Electric field of reflected wave
E_t – Electric field of transmitted wave
H_t – Magnetic field of transmitted wave
H_i – Magnetic field of incident wave
H_r – Magnetic field of reflected wave

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} \quad , \quad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

$$\eta_1 = \frac{E_i}{H_i} \Rightarrow E_i = \eta_1 H_i \quad \rightarrow (1)$$

$$E_r = -\eta_1 H_r \quad \rightarrow (2)$$

$$E_t = \eta_2 H_t \quad \rightarrow (3)$$

According to the boundary condition, the tangential component of E or H is continuous across the boundary.

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$$\therefore H_i + H_r = H_t \rightarrow (4)$$

$$E_i + E_r = E_t \rightarrow (5)$$

$$H_i = \frac{E_i}{\eta_1} \; ; \; H_r = -\frac{E_r}{\eta_1} \; ; \; H_t = \frac{E_t}{\eta_2}$$

From eq (5)

$$\therefore H_t = H_i + H_r = \frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{1}{\eta_1}(E_i - E_r) \rightarrow (6)$$

From eq (4)

$$H_t = \frac{E_t}{\eta_2} = \frac{1}{\eta_2}(E_i + E_r) \rightarrow (7)$$

Equating eq's (6) and (7)

$$\frac{1}{\eta_1}(E_i - E_r) = \frac{1}{\eta_2}(E_i + E_r)$$

$$\eta_2 (E_i - E_r) = \eta_1 (E_i + E_r)$$

$$\eta_2 E_i - \eta_2 E_r = \eta_1 E_i + \eta_1 E_r$$

$$E_i (\eta_2 - \eta_1) = E_r (\eta_1 + \eta_2)$$

Reflection coefficient, $\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$

Also,

$$\frac{E_t}{E_i} = \frac{E_i + E_r}{E_i} = 1 + \frac{E_r}{E_i} \rightarrow (8)$$

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$$\frac{E_t}{E_i} = 1 + \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{\eta_1 + \eta_2 + \eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$\therefore$ Transmission coefficient, $\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2} \rightarrow (9)$

Similarly,

$$\frac{H_r}{H_i} = -\frac{E_r}{E_i} = -\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \rightarrow (10)$$

$$\frac{H_t}{H_i} = \frac{H_i + H_r}{H_i} = 1 + \frac{H_r}{H_i} = 1 + \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}$$

$$\therefore \frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2} \rightarrow (11)$$

Since the permeabilities of perfect dielectric do not differ

$$\mu_1 = \mu_2 = \mu_0$$

$$\eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \; ; \; \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

$$\therefore \frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}} = \frac{\sqrt{\frac{1}{\varepsilon_2}} - \sqrt{\frac{1}{\varepsilon_1}}}{\sqrt{\frac{1}{\varepsilon_1}} + \sqrt{\frac{1}{\varepsilon_2}}}$$

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$$\therefore \quad \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \rightarrow (12)$$

$$\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2} = \frac{\frac{2}{\sqrt{\varepsilon_2}}}{\frac{1}{\sqrt{\varepsilon_1}} + \frac{1}{\sqrt{\varepsilon_2}}} = \frac{2}{\sqrt{\varepsilon_2}}\left[\frac{\sqrt{\varepsilon_1}\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right]$$

$$\frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \rightarrow (13)$$

$$\frac{H_r}{H_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} = \frac{\frac{1}{\sqrt{\varepsilon_1}} - \frac{1}{\sqrt{\varepsilon_2}}}{\frac{1}{\sqrt{\varepsilon_1}} + \frac{1}{\sqrt{\varepsilon_2}}} = \frac{\sqrt{\varepsilon_2} - \sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \rightarrow (14)$$

$$\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1 + \eta_2} = \frac{\frac{2}{\sqrt{\varepsilon_1}}}{\frac{1}{\sqrt{\varepsilon_1}} + \frac{1}{\sqrt{\varepsilon_2}}} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \rightarrow (15)$$

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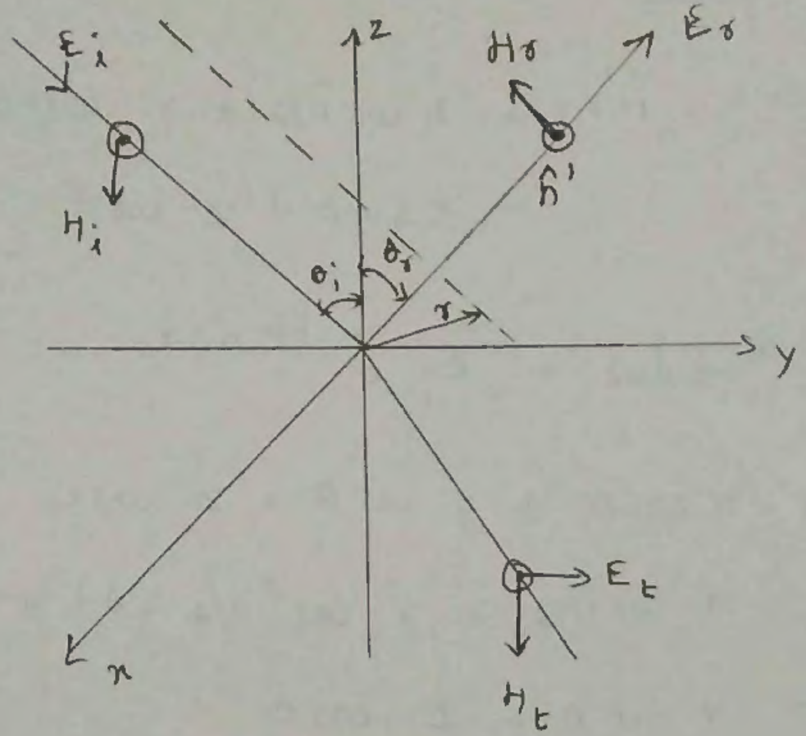
Wave incident obliquely on a perfect conductor:

Case (i): Perpendular (or) Horizontal polarization



"Electric field vector pralled to boundary surface (or) perpendicular to plane of incidence is known as Horizontal polarization".

Let incident and reflected wave makes an angle $\theta_i = \theta_r = \theta$ with z-axis.

$$E_{reflected} = E_r \, e^{-j\beta \hat{n}' \cdot r}$$

$$\hat{n}' \cdot r = x \cos A + y \cos B + z \cos C$$

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E_r is the amplitude of electric field strength of reflected wave

$$E_{reflected} = E_r \, e^{-j\beta(y \sin\theta + z\cos\theta)}$$

where $\hat{n}' \cdot r = x \cos \pi/2 + y \cos(\pi/2 - \theta) + z\cos\theta$

$$= y \sin\theta + z \cos\theta$$

and $E_{incident} = E_i \, e^{-j\beta \hat{n} \cdot r}$

$$\hat{n} \cdot r = x \cos A + y \cos B + z \cos C$$

$$= x \cos \pi/2 + y \cos(\pi/2 - \theta) + z \cos(\pi - \theta)$$

$$= y \sin\theta - z \cos\theta$$

$$\therefore E_{incident} = E_i \, e^{-j\beta(y \sin\theta - z\cos\theta)}$$

But $E_{reflected} = -E_{incident}$

$\therefore$ Total electric field $E_T = E_{incident} + E_{reflected}$

$$E_T = E_i \left[e^{-j\beta(y\sin\theta - z\cos\theta)} - e^{-j\beta(y\sin\theta + z\cos\theta)} \right]$$

$$= E_i \left[e^{-j\beta y \sin\theta} \cdot e^{j\beta z\cos\theta} - e^{-j\beta y\sin\theta} \cdot e^{-j\beta z\cos\theta} \right]$$

$$= E_i \, e^{-j\beta y\sin\theta} \left(e^{j\beta z\cos\theta} - e^{-j\beta z\cos\theta} \right)$$

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$$E_T = E_i \, e^{-j\beta y \sin\theta} \, 2j\sin(\beta z \cos\theta)$$

$$E_T = 2jE_i \sin\beta_z z \cdot e^{-j\beta_y y}$$

where

$$\beta_z = \beta\cos\theta \quad ; \quad \beta_y = \beta\sin\theta$$

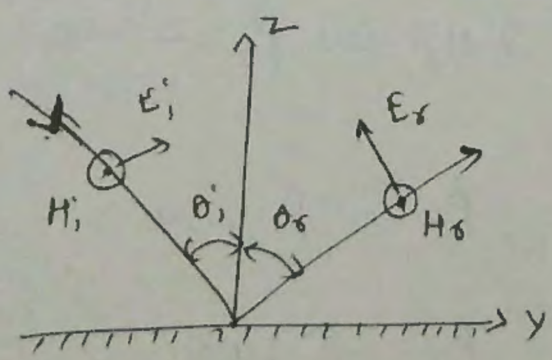
$$\lambda_z = \frac{2\pi}{\beta_z} = \frac{2\pi}{\beta\cos\theta} = \frac{\lambda}{\cos\theta}$$

$$v_z = \frac{\omega}{\beta_z} = \frac{\omega}{\beta\cos\theta} = \frac{\lambda}{\cos\theta}$$

Similarly

$$\lambda_y = \frac{\lambda}{\sin\theta} \quad ; \quad v_y = \frac{\lambda}{\sin\theta}$$

Case (ii) : Parallel (or) vertical polarization :-



vertical polarization

"Electric field vector perpendicular to boundary surface or parallel to plane of incidence is known as parallel or vertical polarization."

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Let the incident and reflected wave H_i and H_r makes an angle $\theta_i = \theta_r = \theta$

$$H_{incident} = H_i e^{-j\beta(y\sin\theta - z\cos\theta)}$$

$$H_{reflected} = H_r e^{-j\beta(y\sin\theta + z\cos\theta)}$$

Since $H_{incident} = H_{reflected}$

$$H_T = H_{incident} + H_{reflected}$$

$$= H_i \left[e^{-j\beta(y\sin\theta - z\cos\theta)} + e^{j\beta(y\sin\theta + z\cos\theta)} \right]$$

$$= H_i e^{-j\beta y\sin\theta} \left(e^{j\beta z\cos\theta} + e^{-j\beta z\cos\theta} \right)$$

$$= 2H_i \cos(\beta z\cos\theta)\left(e^{-j\beta y\sin\theta}\right)$$

$$= 2H_i \cos\beta_z z \; e^{-j\beta_y y}$$

where

$$\beta_z = \beta\cos\theta$$

$$\beta_y = \beta\sin\theta.$$

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Wave incident obliquely on a perfect dielectric :-

Case (i) : Horizontal (or) Perpendicular Polarization:

Electric field E is perpendicular to plane of incidence and parallel to reflecting surface.

$$E_i + E_r = E_t$$

$$1 + \frac{E_r}{E_i} = \frac{E_t}{E_i}$$

But
$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}\, E_t^2 \cos\theta_2}{\sqrt{\varepsilon_1}\, E_i^2 \cos\theta_1}$$

$$= 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \left(\frac{E_r}{E_i}\right)^2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$\left(1 + \frac{E_r}{E_i}\right)\left(1 - \frac{E_r}{E_i}\right) = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \frac{E_r}{E_i} = \frac{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(1 + \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_2}{\cos\theta_1}}{\left(1 + \frac{E_r}{E_i}\right)}$$

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$$= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(1 + \frac{E_r}{E_i}\right) \frac{\cos\theta_2}{\cos\theta_1}$$

$$= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1} + \frac{E_r}{E_i}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1}$$

$$1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1} = \frac{E_r}{E_i}\left(1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1}\right)$$

$$\frac{E_r}{E_i} = \frac{1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \frac{\cos\theta_2}{\cos\theta_1}}{1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_2}} \frac{\cos\theta_2}{\cos\theta_1}} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2}\cos\theta_2}$$

But $\sqrt{\varepsilon_2}\cos\theta_2 = \sqrt{\varepsilon_2(1-\sin^2\theta_2)}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}} \Rightarrow \sin\theta_2 = \frac{\sqrt{\varepsilon_1}\sin\theta_1}{\sqrt{\varepsilon_2}}$$

$$= \sqrt{\varepsilon_2\left(1 - \frac{\varepsilon_1\sin^2\theta_1}{\varepsilon_2}\right)} = \sqrt{\varepsilon_2 - \varepsilon_1\sin^2\theta_1}$$

$$\therefore \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1}\cos\theta_1 - \sqrt{\varepsilon_2 - \varepsilon_1\sin^2\theta_1}}{\sqrt{\varepsilon_1}\cos\theta_1 + \sqrt{\varepsilon_2 - \varepsilon_1\sin^2\theta_1}}$$

$$\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

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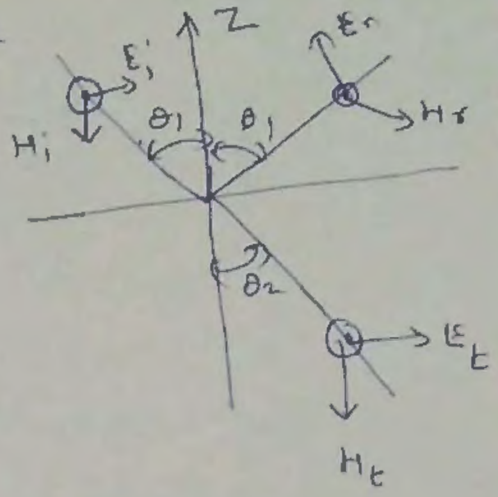
Case (ii) Vertical (or) Parallel polarization :-

From Fig, (E is Prallel to the plane of incidence)

$$(E_i - E_r)\cos\theta_1 = E_t \cos\theta_2$$



$$\frac{E_t}{E_i}\cos\theta_1 = \left(1 - \frac{E_r}{E_i}\right)\cos\theta_1$$

$$\frac{E_t}{E_i} = \left(1 - \frac{E_r}{E_i}\right)\frac{\cos\theta_1}{\cos\theta_2}$$

But $\left(\frac{E_r}{E_i}\right)^2 = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{E_t^2}{E_i^2}\frac{\cos\theta_2}{\cos\theta_1}$

$$\frac{E_r^2}{E_i^2} = 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos^2\theta_1}{\cos^2\theta_2}\frac{\cos\theta_2}{\cos\theta_1}$$

$$= 1 - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$$

$$1 - \frac{E_r^2}{E_i^2} = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$$

$$\left(1 + \frac{E_r}{E_i}\right)\left(1 - \frac{E_r}{E_i}\right) = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\left(1 - \frac{E_r}{E_i}\right)^2 \frac{\cos\theta_1}{\cos\theta_2}$$

$$1 + \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_2}}{\varepsilon_1}\left(1 - \frac{E_r}{E_i}\right)\frac{\cos\theta_1}{\cos\theta_2}$$

$$= \frac{\sqrt{\varepsilon_2}}{\varepsilon_1}\frac{\cos\theta_1}{\cos\theta_2} - \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}\frac{E_r}{E_i}\frac{\cos\theta_1}{\cos\theta_2}$$

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$$\frac{E_r}{E_i}\left(1+\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}\right)=\frac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}-1$$

$$\frac{E_r}{E_i}=\frac{\dfrac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}-1}{1+\dfrac{\sqrt{\varepsilon_2}\cos\theta_1}{\sqrt{\varepsilon_1}\cos\theta_2}}=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\sqrt{\varepsilon_1}\cos\theta_2}{\sqrt{\varepsilon_1}\cos\theta_2+\sqrt{\varepsilon_2}\cos\theta_1}$$

But $\cos\theta_2=\sqrt{1-\sin^2\theta_2}$

$$\therefore \frac{E_r}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\sqrt{\varepsilon_1(1-\sin^2\theta_2)}}{\sqrt{\varepsilon_2}\cos\theta_1+\sqrt{\varepsilon_1(1-\sin^2\theta_2)}}$$

But $\dfrac{\sin\theta_1}{\sin\theta_2}=\dfrac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}$

$$\sin\theta_2=\frac{\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_2}}\sin\theta_1$$

$$\sin^2\theta_2=\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_1$$

$$\therefore \frac{E_r}{E_i}=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\sqrt{\varepsilon_1-\dfrac{\varepsilon_1^2}{\varepsilon_2}\sin^2\theta_1}}{\sqrt{\varepsilon_2}\cos\theta_1+\sqrt{\varepsilon_1-\dfrac{\varepsilon_1^2}{\varepsilon_2}\sin^2\theta_1}}$$

$$=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\dfrac{1}{\sqrt{\varepsilon_2}}\sqrt{\varepsilon_1\varepsilon_2-\varepsilon_1^2\sin^2\theta_1}}{\sqrt{\varepsilon_2}\cos\theta_1+\dfrac{1}{\sqrt{\varepsilon_2}}\sqrt{\varepsilon_1\varepsilon_2-\varepsilon_1^2\sin^2\theta_1}}$$

$$=\frac{\sqrt{\varepsilon_2}\cos\theta_1-\dfrac{\varepsilon_1}{\sqrt{\varepsilon_2}}\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}-\sin^2\theta_1}}{\sqrt{\varepsilon_2}\cos\theta_1+\varepsilon_1/\sqrt{\varepsilon_2}\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}-\sin^2\theta_1}}$$

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Dividing numerator & Denominator by $\frac{\sqrt{\varepsilon_2}}{\varepsilon_1}$

$$\frac{E_r}{E_i} = \frac{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}$$

* Brewster angle:

It is an angle at which no reflection takes place. This occurs when the numerator of above equation is zero

$$\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1} = 0$$

$$\frac{\varepsilon_2}{\varepsilon_1}\cos\theta_1 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}$$

$$\frac{\varepsilon_2}{\varepsilon_1}\sqrt{1-\sin^2\theta_1} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}$$

Squaring on both sides

$$\frac{\varepsilon_2^2}{\varepsilon_1^2}(1-\sin^2\theta_1) = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

$$\frac{\varepsilon_2^2}{\varepsilon_1^2} - \frac{\varepsilon_2^2}{\varepsilon_1^2}\sin^2\theta_1 = \frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1$$

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$$\sin^2\theta_1\left(1-\frac{\varepsilon_2^2}{\varepsilon_1^2}\right)=\frac{\varepsilon_2}{\varepsilon_1}-\frac{\varepsilon_2^2}{\varepsilon_1^2}$$

$$\sin^2\theta_1\left(\varepsilon_1^2-\varepsilon_2^2\right)=\varepsilon_1\varepsilon_2-\varepsilon_2^2$$

$$=\varepsilon_2(\varepsilon_1-\varepsilon_2)$$

$$\sin^2\theta_1=\frac{\varepsilon_2(\varepsilon_1-\varepsilon_2)}{\varepsilon_1^2-\varepsilon_2^2}=\frac{\varepsilon_2(\varepsilon_1-\varepsilon_2)}{(\varepsilon_1+\varepsilon_2)(\varepsilon_1-\varepsilon_2)}$$

$$\sin^2\theta_1=\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2}$$

$$\cos^2\theta_1=\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}$$

$$\therefore\ \tan^2\theta_1=\frac{\varepsilon_2}{\varepsilon_1}$$

$$\tan\theta_1=\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\theta_1=\tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

This is called Brewster angle at which there is no reflected wave when the incident wave is parallel polarized.

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Total internal reflection:

When a wave is incident from denser medium to rarer medium at angle equal to or greater than critical angle, the wave will be totally internally reflected back. This phenomenon is called total internal reflection.

If ε_1 is greater than ε_2 both reflection coefficient for vertical and horizontal polarization become complex.

when

$$\sin\theta_1 > \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

At critical angle, the reflection coefficient has unity value

$$\frac{E_r}{E_i} = \frac{\cos\theta_1 - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}}{\cos\theta_1 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_1}} = 1$$

But $\theta_1 = \theta_c$

$$\cos\theta_c - \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_c} = \cos\theta_c + \sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_c}$$

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$$\sqrt{\frac{\varepsilon_2}{\varepsilon_1} - \sin^2\theta_c} = 0$$

$$\sin^2\theta_c = \frac{\varepsilon_2}{\varepsilon_1}$$

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

$$\boxed{\theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

This is the critical angle at which there is no refracted (transmitted) wave.

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