UNIT - I

Matrices

Two marks:

1) Find the characteristic polynomial of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

Soln: let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

The characteristic polynomial of A is $|A - \lambda I|$.

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix}.$$

$\Rightarrow (1-\lambda)(3-\lambda) - 8$

$\Rightarrow 3 - 4\lambda + \lambda^2 - 8$

$\Rightarrow \lambda^2 - 4\lambda - 5$

2) Find Eigen values and Eigenvectors of the matrix $\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

Soln: let $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$.

The characteristic equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0.$$

$(4-\lambda)(2-\lambda) - 3 = 0.$

$8 - 6\lambda + \lambda^2 - 3 = 0.$

$\lambda^2 - 6\lambda + 5 = 0.$

$$\lambda^2 - 5\lambda - \lambda + 5 = 0$$

$$\lambda(\lambda - 5) - 1(\lambda - 5) = 0$$

$$\lambda = 1, 5$$

$\therefore$ The Eigenvalues of A are $1, 5$.

<u>To find the Eigenvectors</u> :-

Solve $(A - \lambda I) X = 0$

$$\begin{bmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (4-\lambda)x_1 + x_2 &= 0 \\ 3x_1 + (2-\lambda)x_2 &= 0 \end{aligned}\right\} \text{ --- (I)}$$

<u>Case (i)</u> If $\lambda = 1$, then

$$3x_1 + x_2 = 0 \text{ --- (1)}$$

$$3x_1 + x_2 = 0 \text{ --- (2)}$$

(1) & (2) are same.

$$3x_1 = -x_2$$

$$\frac{x_1}{-1} = \frac{x_2}{3}$$

$$\therefore X_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

<u>Case (ii)</u> If $\lambda = 5$ we get

$$-x_1 + x_2 = 0 \text{ --- (3)}$$

$$3x_1 - 3x_2 = 0 \text{ --- (4)}$$

(3) & (4) are same $\frac{x_1}{1} = \frac{x_2}{1}$

$$X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

3) Find ~~all~~ the eigenvalues of $\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$ corresponding to the Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Soln: If the required eigenvalue is λ, then

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\because AX = \lambda X)$$

$$\boxed{2 = \lambda}$$

4) Show that the eigenvalues of a null matrix are zero.

Soln: $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The characteristic equation of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0.$$

$$(0-\lambda)(\lambda^2) = 0$$

$$\lambda^3 = 0.$$

$$\Rightarrow \lambda = 0, 0, 0$$

5) Find the eigenvalues of $2A^2$ of $A = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$. find $4A^2$.

Soln: The characteristic equation of A is $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$.

$$8-6\lambda+\lambda^2-3=0$$

$$\lambda^2-6\lambda+5=0.$$

$$(\lambda-1)(\lambda-5)=0$$

$$\lambda=1,5$$

Eigenvalues of A^2 are $1^2, 5^2$

Eigenvalues of $2A^2$, $2(1^2), 2(5^2)$

$$=2, 50.$$

Eigenvalues of $4A^2 = 4(1^2), 4(5^2).$

$$=4, 4(25)$$

$$=4, 100$$

5) If $X=(-1,0,1)^T$ is the eigen vector of the matrix $A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$. find the Corresponding eigenvalue.

Soln: let $A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 3 \\ 3 & 1 & 1 \end{pmatrix}$.

$X=\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ = Eigenvector.

$(A-\lambda I)X=0.$

$(A - \lambda I) X = 0$.

$$\begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Equations are,

$-(1-\lambda) + 0 + 3 = 0$ — (1)

$-1 + 0 + 1 = 0$ — (2)

$-3 + 0 + (1-\lambda) = 0$ — (3).

(1) $\Rightarrow -1 + \lambda + 3 = 0$

$\Rightarrow +\lambda + 2 = 0$

$\Rightarrow \boxed{\lambda = -2}$

(3) $\Rightarrow -3 + 1 - \lambda = 0$.

$-2 - \lambda = 0$

$-\lambda = 2$

$\boxed{\lambda = -2}$

$\therefore$ ~~Eigenvector~~ Eigenvalue corresponding to the Eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ is $\lambda = -2$.

6) State Cayley Hamilton theorem.

~~So~~ Every square matrix A satisfies its own characteristic equation, ie)

$$A^n - C_1 A^{n-1} + \dots + (-1)^{n-1} C_{n-1} A + (-1)^n C_n I = 0.$$

7) Write the uses of Cayley Hamilton theorem.

To Calculate

(i) the positive integral powers of A

(ii) the inverse of a non-singular matrix A.

8). Prove that $A = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix}$ is an orthogonal.

Soln: $AA^T = \begin{pmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{pmatrix} \begin{pmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{pmatrix}$

$= \begin{pmatrix} 4/9 + 1/9 + 4/9 & -4/9 + 2/3 + 2/3 & -2/9 - 2/9 + 4/9 \\ -4/9 + 2/9 + 2/9 & 4/9 + 4/9 + 1/9 & 2/9 + 2/9 - 4/9 \\ -2/9 - 2/9 + 4/9 & 2/9 - 4/9 + 2/9 & 1/9 + 4/9 + 4/9 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$

Similarly, $A^T A = I$.

$\therefore$ A is orthogonal.

9) If the modal matrix is $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$ then what is the normalised modal matrix?

Soln:- $\begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$.

10) Write the matrix of the following quadratic forms.

(i) $x_1^2+3x_2^2+3x_3^2-2x_2x_3$

(ii) $2x_1^2+4x_3^2-2x_2^2+2x_1x_2-6x_1x_3+6x_2x_3$.

(iii) $2x^2+8z^2+4xy+10zx-2yz$.

Soln:-

The matrix corresponding to the quadratic form is.

$$\begin{pmatrix} \text{Coeff of } x_1^2 & \frac{1}{2}\text{Coeff of } x_1x_2 & \frac{1}{2}\text{Coeff of } x_1x_3 \\ \frac{1}{2}\text{Coeff of } x_2x_1 & \text{Coeff, of } x_2^2 & \frac{1}{2}\text{Coeff of } x_2x_3 \\ \frac{1}{2}\text{Coeff of } x_3x_1 & \frac{1}{2}\text{Coeff, } x_3x_2 & \text{Coeff, } x_3^2 \end{pmatrix}$$

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

(ii) $\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{pmatrix}$.

11) Write the quadratic form of the following matrices.

(i) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -2 & -4 \\ 3 & -4 & -3 \end{pmatrix}$.

Ans:-

The Q.F is, $x^2 - 2y^2 - 3z^2 + 4xy + 6xz - 8yz$.

~~ii) $y^2 + 3z^2 - 2$~~

ii) $\begin{pmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{pmatrix}$.

Ans:- The Q.F is

$y^2 + 3z^2 - 2xy + 4xz + 8yz$.

(iii) $\begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & -2 \\ 3 & -2 & 0 \end{pmatrix}$

The Q.F is,

$$2xy + 6xz - 4yz.$$

12) Discuss the nature of the following quadratic forms.

(i) $11x_1^2 + 2x_2^2 + 2x_3^2 + 4x_1x_2 - 2x_2x_3 + 4x_1x_3$.

Soln:-

The matrix form is

$$A = \begin{pmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}$$

$$D_1 = |11| > 0$$

$$D_2 = \begin{vmatrix} 11 & 2 \\ 2 & 2 \end{vmatrix} = 22 - 4 = 18 > 0.$$

$$D_3 = \begin{vmatrix} 11 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = 11(4+1) - 2(4+2) + 2(-2-4)$$

$$= 55 - 12 - 12$$

$$= 31 > 0.$$

Since all the values are positive, the given quadratic form is positive definite.

(iii). $2x_1^2+2x_1x_3+3x_3^2$.

Soln: $A=\begin{pmatrix}2&0&1\\0&0&0\\1&0&3\end{pmatrix}$

$D_1=|2|>0$

$D_2=\begin{vmatrix}2&0\\0&0\end{vmatrix}=0$

$D_3=|A|=2(0-0)-0(\ )+1(0)$
$=0$.

$\therefore$ The Quadratic form is positive semidefinite.

21/4/21

13) If 3 and 5 are the two eigen values of the matrix $A=\begin{pmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{pmatrix}$ then find its third eigen value and hence $|A|$.

Soln: WKT,
Sum of the eigen values
= Sum of the main diagonal elements of A
$=8+7+3$

$\lambda_1+\lambda_2+\lambda_3=18$.

$3+5+\lambda_3=18 \Rightarrow \lambda_3=18-8=10$.

The third eigen value is 10.

$|A| =$ Product of the eigen values

$= 3 \times 5 \times 10$

$= 150.$

14) If λ is the eigenValue of A then prove that λ^2 is the eigenvalue of A^2

Proof: let λ_i be the eigenvalue of A and X_i the corresponding eigenvector.

Then $AX_i = \lambda_i X_i$

we have, $A^2 X_i = A(AX_i) = A(\lambda_i X_i)$
$= \lambda_i (A(X_i))$
$= \lambda_i (\lambda_i X_i)$
$= \lambda_i^2 X_i$

Hence, λ_i^2 is an eigenvalue of A^2

Thus If λ is the eigenvalue of A, then λ^2 is the eigenvalue of A^2.

15) If the eigenvalues of the matrix A of order 3x3 are 2, 3, 1, then find the determinant of A.

Soln: $|A| =$ product of the eigenvalues.
$= 2 \times 3 \times 1 = 6.$

16) The product of two eigen values of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. find the third eigen value.

Soln: We know that $|A| =$ Product of the eigen values.

$6(9-1) + 2(-6+2) + 2(+2-6) = (16)\lambda_3.$

$16(\lambda_3) = 32$

$\Rightarrow \boxed{\lambda_3 = 2}$

17) Identify the nature, index and signature of the quadratic form ~~whose Canon~~ q: $2x_1x_2 + 2x_2x_3 + 2x_3x_1$

Soln: The matrix of the quadratic form is given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The characteristic equation is

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

S_1 = Sum of the main diagonal elements $= 0$.

S_2 = Sum of the minors of the main diagonal element

$= (0-1) + (0-1) + (0-1) = -3.$

$S_3 = |A| = 2.$

$\therefore \lambda^3 - 3\lambda - 2 = 0.$

$(\lambda+1)^2(\lambda-2) = 0.$

$\lambda = -1, -1, 2.$

Nature : Indefinite

Rank = 3 = Number of eigen values = (r)

Index = 1 = Number of positive eigenvalue = p.

Signature $= 2(1) - 3 = -1$
(s)
$= 2(p) - r = -1.$

Unit - I

NOV - DEC - 2020

1. Give that α, β are the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$. form the matrix whose eigenvalues are α^2, β^2. (2)

2. If the Canonical form in three variables u, v, w is given by $3v^2 + 15w^2$ corresponding to a quadratic form, then state the nature, index, signature and rank of the quadratic form (2)

3) The eigenvalues of a real symmetric matrix A corresponding to the eigenvalues 2, 3, 6 are respectively $(1\ 0\ -1)^T$, $(1\ 1\ 1)^T$ and $(-1\ 2\ -1)^T$ (8)

4) Show that A satisfies its own characteristic equation and hence find A^6 if $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ (8)

5) Using Cayley Hamilton theorem. find the inverse of the matrix.

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}. \quad (8)$$

6) Reduce the quadratic form $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$ into a Canonical form using an orthogonal transf

April/May - 2019

1) If λ is the eigenvalue of the matrix A, then prove that λ^2 is the eigenvalue of the A^2. (2)

2) If the eigenvalue of the matrix A of order 3x3 are 2, 3 and 1, then find the determinant of A. (2)

3) Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ (8)

4) Using Cayley Hamilton theorem find A^{-1} if $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$ (8)

5) Reduce the quadratic form $2xy - 2yz + 2xz$ into a canonical form by an orthogonal reduction. (16)

April/May - 2018

1) If 3 and 5 are two eigenvalues of the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. then find its third eigen value and hence $|A|$. (2)

2) Show that the eigenvalues of a null matrix are zero (2)

3) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$$ (8)

4) Using Cayley Hamilton theorem find the inverse of the given matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$ (8)

5) Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to a canonical form through an orthogonal transformation. Find also its nature (16).

January - 2022

1. If 2, −1, −3 are the eigenvalues of a matrix "A", then find the eigenvalues of the matrix $A^2 - 2I$.

2. Write down the matrix for the following quadratic form. 54·3

$$2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$$

3. Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & -3 \end{pmatrix}$.

4. Using Cayley Hamilton theorem find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

5. Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ to canonical form through an orthogonal transformation. Also find its nature, rank index and signature.

April / May - 2022

1. If $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$ then find the eigenvalues of A^{-1} (S4 Q)

2. Prove that $x^2 - y^2 + 4z^2 + 4xy + 2yz + 6xz$ is indefinite. (S4 5)

3. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

4. Using Cayley Hamilton theorem find A^4 if $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{pmatrix}$.

5. Reduce the quadratic form $6x^2+3y^2+3z^2-4xy-2yz+4xz$ into a Canonical form through an orthogonal reduction.

Nov/Dec - 2022 :

1. The eigenvalues and the corresponding eigenvectors of a 2×2 matrix is given by $\lambda_1 = 8$; $x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 4$; $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ find the corresponding matrix.

2. Determine the nature, index and signature of the quadratic form $x_1^2+5x_2^2+x_3^2+2x_2x_3+6x_3x_1+2x_1x_2$

3. Obtain an orthogonal transformation which will transform the

quadratic form

$Q = 2x_2x_3 + 2x_3x_1 + 2x_1x_2$ to

Canonical form.

4. An elastic membrane in the x_1x_2-plane with boundary circle $x_1^2 + x_2^2 = 1$ is stretched so that a point $P = (x_1, x_2)$ goes over a point $Q = (y_1, y_2)$ given by $y_1 = 5x_1 + 3x_2$ and $y_2 = 3x_1 + 5x_2$. Find the principal directions that is, the directions of the position vector x of P for which the direction of the position vector y of Q is the same or exactly opposite. What shape does the boundary circle take under this deformation?

Unit - II

Two marks:

1. Define a function with an example.

A function f is a rule that assign to each element x in a set D exactly one element, called $f(x)$, in a set E.

2. Find the domain and range of $f(x) = 2x-1$.

Soln: Since the expression $2x-1$ is defined for all real numbers,

the domain of f is the set of all real numbers and range of f is also the set of all real numbers.

$\therefore$ The domain and range of f is R.

3. Find the domain of the function $f(x) = \frac{1}{x^2-x}$.

Soln: The given function can be written as $f(x) = \frac{1}{x(x-1)}$ and the function f is undefined at $x=0$ and $x=1$. The domain of f is $\{x / x \neq 0, x \neq 1\}$ which is written as $(-\infty, 0) \cup (0,1) \cup (1,\infty)$

4. Determine whether the given function $f(x) = x^5 + x$ is even or odd?

Soln: $f(-x) = (-x)^5 + (-x)$

$= -x^5 - x$

$= -(x^5 + x)$

$= -f(x)$.

$\therefore f$ is an odd function.

5. Verify whether the given function $f(x) = 1 - x^4$ is an odd or even?

Soln: $f(-x) = 1 - (-x)^4$

$= 1 - x^4$

$= f(x)$

$\therefore f(x)$ is an even function.

6) Evaluate the difference quotient for the function

Soln:

let $f(x) = 4 + 3x - x^2$

$f(3+h) = 4 + 3(3+h) - (3+h)^2$

$= 4 + 3(3+h) - (9 + h^2 + 6h)$

$= 4 + 9 + 3h - 9 - h^2 - 6h$

$= 4 - 3h - h^2$

$f(3) = 4 + 3(3) - (3)^2$

$= 4 + 9 - 9$

$= 4$

$$\frac{f(3+h) - f(3)}{h} = \frac{1}{h}\left(1 - 3h - h^2 - 4\right)$$

$$= \frac{1}{h}\left(-3h - h^2\right)$$

$$= -(3+h).$$

7) Determine the limit

$$\lim_{x \to 5} (2x^5 - 3x + 4)$$

Soln: $\lim_{x \to 5} (2x^2 - 3x + 4)$

$$= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} 3x + \lim_{x \to 5} (4)$$

$$= 2 \lim_{x \to 5} (x^2) - 3 \lim_{x \to 5} x + \lim_{x \to 5} (4)$$

$$= 2(5)^2 - 3(5) + 4 = 39.$$

8) State squeez or sandwich theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a).

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$,

then, $\lim_{x \to a} g(x) = L$.

The squeez theorem, is otherwise called as sandwich theorem or the pinching theorem. It states that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

9. State the intermediate value theorem.

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exist number c in (a,b) such that $f(c)=N$.

10) Find the derivative of

$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}$$

Soln: $f(x) = (x^3+x+1)^{-1/3}$

$$f'(x) = -\frac{1}{3}(x^2+x+1)^{-4/3} \frac{d}{dx}(x^2+x+1)$$

$$= -\frac{1}{3}(x^2+x+1)^{-4/3}(2x+1)$$

11) Find the ~~the~~ differentiation of $y = \frac{1}{\sin^{-1}(x)}$.

Soln:-

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1}$$

$$= -1(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= \frac{-1}{(\sin^{-1}x)^2} \cdot \frac{1}{\sqrt{1-x^2}}$$

12. Differentiate $f(x) = \log_{10}(2+\sin x)$

$$f'(x) = \frac{d}{dx}(\log_{10}(2+\sin x))$$

$$= \frac{1}{2+\sin x}\frac{d}{dx}(2+\sin x)$$

$$= \frac{\cos x}{(2+\sin x)}$$

13. State the Rolle's theorem.

Let f be a function that satisfies the following three assumptions.

(i) f is continuous on the closed interval $[a,b]$.

(ii) f is differentiable on the open interval (a,b).

(iii) $f(a) = f(b)$.

Then there is a number c in (a,b) such that $f'(c) = 0$.

1. Define the Critical number in maximum or minimum values.

If f has a local maximum or minimum at C, then C is a Critical number of f.

15 Define an inflection point.

A point P on a Curve $y=f(x)$ is Called an inflection point if f is Continuous and the Curve Changes from Concave upward to

16. Evaluate $\lim\limits_{x\to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2}$.

Soln: $\lim\limits_{x\to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim\limits_{x\to \pi/2} \frac{2\cos^2 x}{4\left(\pi/2 - x\right)^2}$

$= \frac{1}{2} \lim\limits_{x\to \pi/2} \left(\frac{\sin\left(\pi/2 - x\right)}{\pi/2 - x}\right)^2$

$= \frac{1}{2}$ $\quad \left(\because \lim\limits_{y\to 0} \frac{\sin y}{y} = 1\right)$

17) Find $\frac{d}{dx}\left((\sin x)^{\cos x}\right)$

$\frac{d}{dx}\left(\sin x^{\cos x}\right) = \frac{d}{dx}\left(e^{\log (\sin x)^{\cos x}}\right)$

$= \frac{d}{dx}\left(e^{\cos x \log(\sin x)}\right)$

$= (\cos x \log \sin x)\left(\frac{\cos x}{\sin x} \times \cos x + \log \sin x \times (-\sin x)\right)$

$= (\cos x \log \sin x)\left(\frac{\cos^2 x}{\sin x} + \left(-\sin x \times \log \sin x\right)\right)$

18) Show that $\lim_{x\to 0} x^2 \sin(1/x) = 0$.

Soln:

We have $-1 \le \sin(1/x) \le 1$,

hence $-x^2 \le x^2 \sin(1/x) \le x^2$

and $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$.

By squeez theorem,

$\lim_{x\to 0} x^2 \sin(1/x) = 0$.

19) If $f(1) = 10$, $f'(x) \geq 2$ for $1 \le x \le 4$. how small can $f(4)$ possibly be?

Soln: By mean value theorem

$$\frac{f(4) - f(1)}{4-1} = f'(c)$$

$$\frac{f(4) - (10)}{3} \geq 2$$

$$f(4) \geq 6 + 10$$

$$f(4) \geq 16.$$

The minimum value is 16.

). Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?

Soln:

The horizontal tangents occur where the slope $\frac{dy}{dx}$ is zero.

$\frac{dy}{dx} = 4x^3 - 4x$

$\Rightarrow 4x(x^2 - 1) = 0$

$x = 0$ or $x = \pm 1$.

$\therefore$ The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$ and -1. The corresponding points on the curve are

$x = 0, \quad y = 2 \Rightarrow (0, 2)$

$x = 1, \quad y = 1 \Rightarrow (1, 1)$

$x = -1, \quad y \; 1 \Rightarrow (-1, 1)$

Unit - II

University Questions

NOV-DEC-2020

1) If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, then find $\lim_{x\to 1} f(x)$. (2)

2) If $xe^y = x-y$, then find find $\frac{dy}{dx}$ by implicit differentiation. (2)

3) Use the intermediate value theorem to show that there is a root of the equation $\sqrt[3]{x} = 1-x$ in the ~~in the~~ interval $(0,1)$. (8)

4) Show that the function $f(x) = |x-6|$ is not differentiable at 6. Find a formula for first derivative of f and sketch its graph. (8)

5) Find the equation of the tangent line to the curve $y = x^4 + 2x^2 - x$ at the point $(1,2)$. (8)

6) Find the local maximum value, local minimum value, the interval of concavity and the inflection points of a function $f(x) = x^3 - 3x^2 - 12x$. Also sketch the graph of f that satisfies all the above conditions. (8)

April may - 2019

1. Check whether $\lim_{x \to -3} \frac{3x+9}{|x+3|}$ exist (2).
2. Find the Critical points of $y = 5x^3 - 6x$ (2)
3. find $\frac{dy}{dx}$ if $y = x^2 e^{2x}(x^2+1)^4$. (8)
4. For what value of the constant b, is the function f continuous on $(-\infty, \infty)$ if $f(x) = \begin{cases} bx^2 + 2x & \text{if } x < 2 \\ x^3 - bx & \text{if } x \geq 2 \end{cases}$ (8)
5. If $f(x) = 2x^3 + 3x^2 - 36x$, find the intervals on which it is increasing or decreasing, the local maximum and local minimum values of $f(x)$. (16)

Jan - 2018

1. Sketch the graph of the function $f(x) = \begin{cases} 1+x, & x < -1 \\ x^2, & -1 \leq x \leq 1 \\ 2-x, & x \geq 1 \end{cases}$ and use it to determine the value of "a" for which $\lim_{x \to a} f(x)$ exists. (2)
2. Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where. (2)
3) For what value of the constant "c" is the function "f" continuous on $(-\infty, \infty)$. (8)

$$f(x) = \begin{cases} cx^2 + 2x; & x < 2 \\ x^3 - cx; & x \geq 2 \end{cases} \quad (8)$$

4) Find the local maximum and minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both the first and second derivative tests. (8)

5) Find y'', if $x^4 + y^4 = 16$. (8)

6) Find the tangent line to the equation $x^3 + y^3 = 6xy$ at the point $(3,3)$ and at what point the tangent line horizontal in the first quadrant. (8)

NOV-DEC-2018

1. Find the domain of $f(x) = \sqrt{3-x} - \sqrt{2+x}$. (2)

2. Evaluate $\lim\limits_{t \to 1} \dfrac{t^4 - 1}{t^3 - 1}$. (2)

3. Guess the value of the limit (if exists) for the function $\lim\limits_{x \to 0} \dfrac{e^{5x} - 1}{x}$ by evaluating the function at the given numbers $x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001$ (correct to six decimal places). (6)

4) For the function $f(x) = 2+2x^2-x^4$ find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and the inflection points. (10)

5) Find the values of a and b that makes f continuous on $(-\infty,\infty)$.

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2}, & \text{if } x<2 \\ ax^2-bx+3, & \text{if } 2 \le x < 3 \\ 2x-a+b, & \text{if } x \geq 3 \end{cases} \quad (8)$$

6) Find the derivative of $f(x) = \cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$ (4)

7) Find y' for $\cos(xy) = 1+\sin y$. (4)

Ans:- (NOV DEC 2018 (3) Soln)

The limit of the following function is estimated as x approaches zero from both sides.

$$\lim_{x\to 0} \frac{e^{5x}-1}{x}, \quad x = \pm 0.5, \pm 0.1, \pm 0.01, \pm 0.001, \pm 0.0001.$$

The limit of the following function is estimated as x approaches zero from both sides.

$$f(x) = \frac{e^{5x}-1}{x}$$

$$f(0.5) = \frac{e^{5(0.5)}-1}{0.5} = 22.365000$$

$$f(-0.5) = \frac{e^{5(-0.5)}-1}{-0.5} = 1.835830$$

$$f(0.1) = \frac{e^{5(0.1)}-1}{0.1} = 6.487210$$

$$f(-0.1) = \frac{e^{5(-0.1)}-1}{-0.1} = 3.934690$$

$$f(0.01) = \frac{e^{5(0.01)}-1}{+0.01} = 5.127110$$

$$f(-0.01) = \frac{e^{5(-0.01)}-1}{-0.01} = 4.877060$$

$$f(0.001) = \frac{e^{5(0.001)}-1}{0.001} = 5.012520$$

$$f(-0.001) = \frac{e^{5(-0.001)}-1}{-0.001} = 4.987520$$

$$f(0.0001) = \frac{e^{5(0.0001)}-1}{0.0001} = 5.001250$$

$$f(-0.0001) = \frac{e^{5(-0.0001)}-1}{-0.0001} = 4.998750$$

$\therefore$ The function is approaching 5 as x approaches 0 from both sides $\lim_{x\to 0} \frac{e^{5x}-1}{x} = 5$.

Eg 1.3. For the function $f(x) = 2 + 2x^2 - x^4$. Find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and the inflection points.

Soln: $f(x) = 2 + 2x^2 - x^4$

$$f'(x) = 4x - 4x^3$$
$$= 4x(1 - x^2)$$
$$= -4x(x^2 - 1)$$
$$= -4x(x+1)(x-1)$$

The critical points are $0, 1, -1$.

Interval	x	$(x+1)$	$(x-1)$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$+$	increasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$+$	$-$	$-$	decreasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$+$	creasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$-$	increasing on $(1, \infty)$

f' Changes from positive to negative at $x=-1$, f has a local maximum at $x=-1$, $f(-1) = 2 + 2(-1)^2 - (-1)^4$
$= 4 - 1 = 3.$

f' Changes from negative to positive at $x=0$. f has a local minimum at $x=0$, $f(0) = 2 + 2(0)^2 - (0)^4$
$= 2.$

f' changes from positive to negative at $x=1$, f has a local maximum at $x=1$, $f(1) = 2 + 2(1)^2 - (1)^4$
$= 4 - 1 = 3.$

$f'(x) = +4x - 4x^3.$

$f''(x) = 4 - 12x^2$

$= 4(1 - 3x^2)$

Let $f''(x) = 0$, $1 - 3x^2 = 0$

$1 = 3x^2$

$x^2 = 1/3$

$x = \pm\sqrt{1/3}.$

Interval	$f''(x)$	Concativity
$x < -\sqrt{1/3}$	$-$	Concave downward
$-\sqrt{1/3} < x < \sqrt{1/3}$	$+$	Concave upward
$x > \sqrt{1/3}$	$-$	Concave downward.

The Curve Changes from Concave downward to upward at $x = -\sqrt{\frac{1}{3}}$

The inflection point is $\left(-\frac{1}{\sqrt{3}}, f\left(-\frac{1}{\sqrt{3}}\right)\right)$.

The Curve Changes from Concave upward to downward at $x = \sqrt{\frac{1}{3}}$

The inflection point is $\left(\frac{1}{\sqrt{3}}, f\left(\frac{1}{\sqrt{3}}\right)\right)$

$$f\left(-\frac{1}{\sqrt{3}}\right) = 2 + 2\left(-\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^4$$

$$= 2 + \frac{2}{3} - \frac{1}{9}$$

$$= \frac{18+6-1}{9} = \frac{23}{9}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 2 + 2\left(\frac{1}{\sqrt{3}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^4$$

$$= 2 + \frac{2}{3} - \frac{1}{9}$$

$$= \frac{18+6-1}{9} = \frac{23}{9}.$$

The inflection points are $\left(-\frac{1}{\sqrt{3}}, \frac{23}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{23}{9}\right)$

Ex:-4 For the function $f(x) = x^3 - 3x^2 - 12x$, find the local maximum and local minimum value and the intervals of Concavity and the inflection points of a function

Also sketch the graph of f that satisfies all the above conditions.

Soln:

$$f(x) = x^3 - 3x^2 - 12x.$$

$$f'(x) = 3x^2 - 6x - 12$$

$$= 3(x^2 - 2x - 4)$$

$$= 3(x^2 - 2x - 4)$$

Critical numbers of f(x) are

$$f'(x) = 0$$

$$3(x^2 - 2x - 4) = 0$$

$$x^2 - 2x - 4 = 0$$

$$x = \frac{+2 \pm \sqrt{4 - 4(1)(-4)}}{2(1)}$$

$$= \frac{+2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2}$$

$$= 1 \pm \sqrt{5}.$$

We divide the real line into intervals whose end points are the critical numbers $x = 1+\sqrt{5}, 1-\sqrt{5}$ and list them in the table.

Interval	$(x-(1+\sqrt{5}))$	$(x-(1-\sqrt{5}))$	$f'(x)$	$f(x)$
$x < 1-\sqrt{5}$	$-$	$-$	$+$	increasing
$1-\sqrt{5} < x < 1+\sqrt{5}$	$-$	$+$	$-$	decreasing
$x > 1+\sqrt{5}$	$+$	$+$	$+$	increasing.

$f'(x)$ changes from positive to negative at $x = 1-\sqrt{5}$

f has a local maximum at $x = 1-\sqrt{5}$

$$f(1-\sqrt{5}) = (1-\sqrt{5})^3 - 3(1-\sqrt{5})^2 - 12(1-\sqrt{5})$$

$$= (1 - 3\sqrt{5} + 15 - 5\sqrt{5}) - 3(1+5-2\sqrt{5}) - 12 + 12\sqrt{5}$$

$$= -14 + 10\sqrt{5}$$

$f'(x)$ Changes from negative to positive at $x = 1+\sqrt{5}$.

Thus the function has a local minimum at $x = 1+\sqrt{5}$ and local minimum value is

$$f(1+\sqrt{5}) = (1+\sqrt{5})^3 - 3(1+\sqrt{5})^2 - 12(1+\sqrt{5})$$

$$= 1 + 3\sqrt{5} + 15 + 5\sqrt{5} - 3 - 15 - 6\sqrt{5} - 12 - 12\sqrt{5}$$

$$= -14 - 10\sqrt{5}.$$

For Concavity, $f''(x) = 0$.

$$f'(x) = 3x^2 - 6x - 12$$

$$f''(x) = 6x - 6 = 6(x-1)$$

$$f''(x) = 0 \Rightarrow 6x - 6 = 0$$

$$\boxed{x = 1}$$

Interval	$f''(x)$	Concavity
$x < 1$	$-$	Concave downwards.
$x > 1$	$+$	Concave upwards.

The curve changes from concave downward to upward at $x=1$.

The inflection point is $(1, f(1))$

$f(1) = (1)^3 - 3(1)^2 - 12(1)$

$= 1 - 3 - 12$

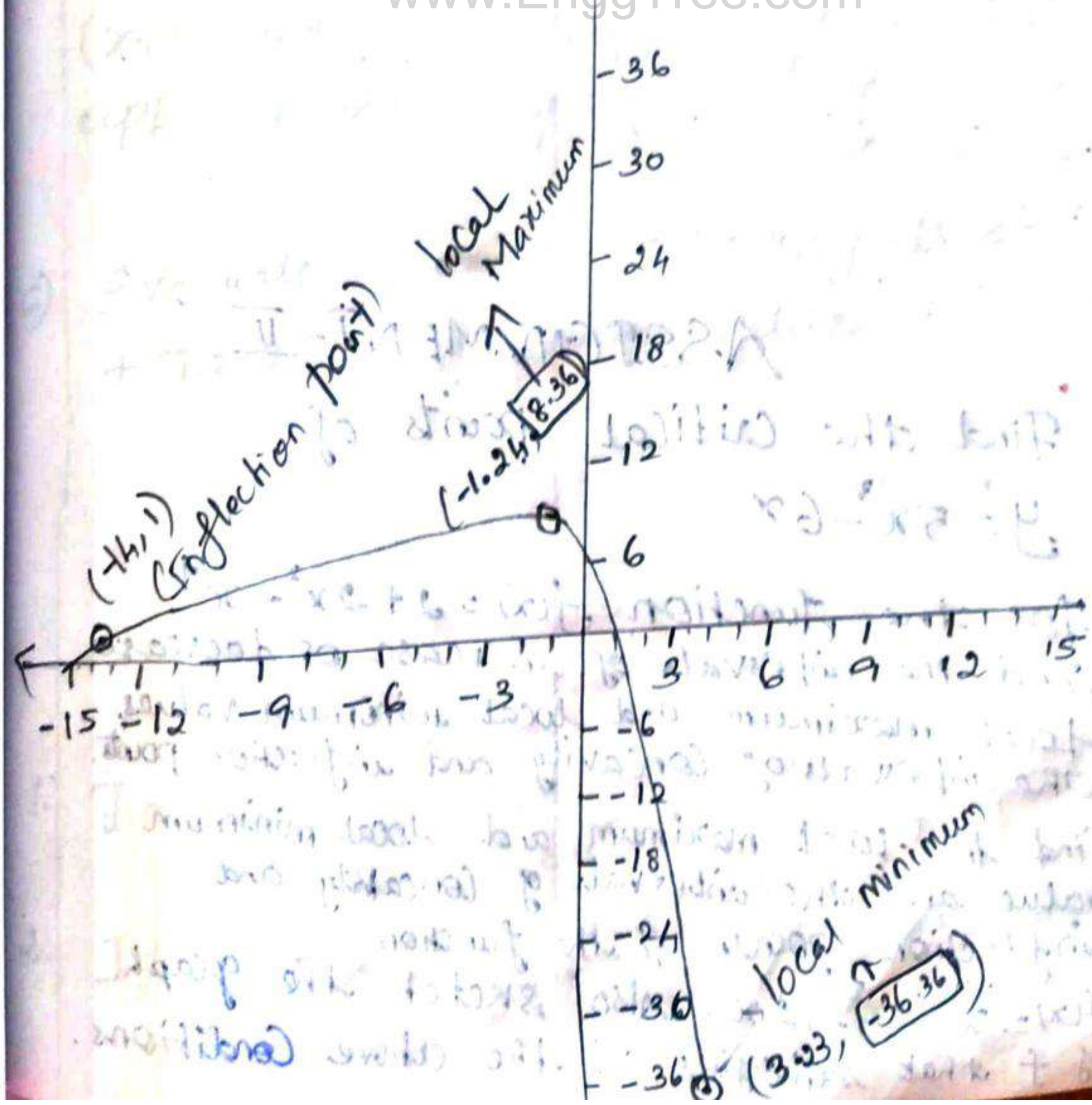
$= -14$.

The points of inflection is $(1, -14)$.

Graph

$f(x) = x^3 - 3x^2 - 12x$

x	-14	$1+\sqrt{5}$ (3.23)	$1-\sqrt{5}$ (-1.24)
$f(x)=y$	1	$-14-10\sqrt{5}$ (-36.36)	$-14+10\sqrt{5}$ (8.36)



Unit - II

April/May - 2022

1) Evaluate $\lim_{x \to 5} (2x^2 - 3x + 4)$

2) Find the domain of the function
$f(x) = \frac{2x^3 - 5}{x^2 + x - 6}$.

3) For what values of a and b,
is $f(x) = \begin{cases} -2, & x \le -1 \\ ax - b, & -1 \le x < 1 \\ 3, & x \ge 1 \end{cases}$
Continuous at every x?

4) Find the differential coefficients of $\frac{(a-x)^2 (b-x)^3}{(c-2x)^3}$.

5) Evaluate (1) $\frac{d}{dx}(3x^5 \log x)$
(2) $\frac{d}{dx}\left(\frac{x^3}{3x-2}\right)$

6) Find the maximum and minimum values of $2x^3 - 3x^2 - 36x + 10$.

January 2022

1) Find the domain of the function $f(x) = \frac{2x^3 - 5}{x^2 + x - 6}$

2) Evaluate $\lim_{x \to 1} \frac{x^2 - 4x}{x^2 - 3x - 4}$

3) If $x^2 + y^2 = 25$, then find $\frac{dy}{dx}$ and also find an equation of the tangent line to the curve $x^2 + y^2 = 25$ at the point $(3, 4)$

4) If $f(x) = xe^x$, then find $f'(x)$. Also find the nth derivative $f^n(x)$.

5) Differentiate the function $f(x) = \frac{\sec x}{1 + \tan x}$. for what values of x, the graph of $f(x)$ has a horizontal tangent?

6) Find the absolute maximum and absolute minimum values of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ on the interval $[-2, 3]$.

NOV/DEC - 2022

1. For what values of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x ; & x < 2 \\ x^3 - cx ; & x \geq 2 \end{cases}$$

2. Find the slope of the circle $x^2 + y^2 = 25$ at $(3, -4)$.

3. Find y'' if $x^4 + y^4 = 16$.

4. Differentiate $y = (2x+1)^5 (x^3 - x + 1)^4$.

5. Find the intervals on which $f(x) = -x^3 + 12x + 5$, $-3 \leq x \leq 3$ is increasing and decreasing. Where does the function assume extreme values? What are those values?

$\frac{x(x+4)}{(x-4)(x+1)}$ -4 1 -3

Unit - III

If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Soln:

$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}$, $\frac{\partial v}{\partial y} = -\frac{zx}{y^2}$, $\frac{\partial w}{\partial z} = -\frac{xy}{z^2}$

$\frac{\partial u}{\partial y} = z/x$, $\frac{\partial v}{\partial x} = z/y$, $\frac{\partial w}{\partial x} = y/x$

$\frac{\partial u}{\partial z} = y/x$, $\frac{\partial v}{\partial z} = x/y$, $\frac{\partial w}{\partial y} = x/z$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$$= \begin{vmatrix} -yz/x^2 & z/x & y/x \\ z/y & -zx/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= \frac{-yz}{x^2}\left(\frac{zx^2y}{y^2z} - x^2/yz\right)$$

$$-z/x\left(\frac{-xyz}{yz^2} - \frac{xy}{yz}\right) + y/x\left(\frac{xz}{yz} + \frac{xyz}{y^2z}\right)$$

$$= 0 - z/x\left(-2x/z\right) + y/x\left(2x/y\right)$$

$$= 2+2 = 4.$$

2. Find the Jacobian of transformation
$x = u(1-v)$, $y = uv$

Soln:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = u(1-v)$; $y = uv$

$\frac{\partial x}{\partial u} = 1-v$, $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = -u$, $\frac{\partial y}{\partial v} = u$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u(1-v) + uv$

$= u - uv + uv = u.$

3. Find $\frac{du}{dt}$ where $u = \sin\left(\frac{x}{y}\right)$.

Soln: $u = \sin\left(\frac{x}{y}\right)$. $x = e^t$, $y = t^2$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}.$$

$$\frac{\partial u}{\partial x} = \cos\left(\frac{x}{y}\right) \cdot \left(\frac{1}{y}\right)$$

$$\frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right) \left(-\frac{x}{y^2}\right)$$

$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2t$

$\frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t + \cos\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)(2t)$

$= \frac{e^t}{y}\cos\left(\frac{x}{y}\right) - \frac{2xt}{y^2}\cos\left(\frac{x}{y}\right)$

$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{2e^t t}{t^2}\cos\left(\frac{e^t}{t^2}\right).$

4. What is the total differential of u

The total differential of a homogeneous function u is given by $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

5. If $u = x^2, v = y^2$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln:

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$= \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy.$

6. If $u = \frac{y^2}{x}$ and $v = \frac{x^2}{y}$, find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

$$= \begin{vmatrix} -y^2/x^2 & 2y/x \\ 2x/y & -x^2/y^2 \end{vmatrix}$$

$$= 1 - 4$$

$$= -3$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}.$$

7. Define stationary points?

Stationary points are the points at which function attains its maximum or minimum value. The points can be obtained by solving the equations.

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0.$$

8. Define Saddle points.

Saddle points are the points at which function attains neither maximum nor minimum.

9. Find the stationary points of $f(x,y) = x^2 - xy + y^2 - 2x + y$.

Soln: $\frac{\partial f}{\partial x} = 2x - y - 2 = 0$

$\frac{\partial f}{\partial y} = -x + 2y + 1 = 0.$

$2x - y - 2 \Rightarrow 4x - 2y = 4$

$x - 2y = 1 \Rightarrow \underline{x - 2y = 1}$

$3x = 3$

$x = 1$

Put $x=1$ in $x-2y=1$ we get

$1 - 2y = 1$

$2y = 1 - 1$

$2y = 0$

$\Rightarrow y = 0$

The Stationary point is $(1,0)$.

10) Find the Taylor's series expansion of x^y near the point $(1,1)$ upto the first degree term.

Soln: Given $f(x,y) = x^y$; $f(1,1) = 1$.

$f_x(x,y) = yx^{y-1}$; $f_x(1,1) = 1$

$f_y(x,y) = x^y \log x$; $f_y(1,1) = 0$.

The required series is

$f(x,y) = f(1,1) + \frac{1}{1!}\left((x-1) f_x(1,1) + (y-1) f_y(1,1)\right) + \ldots$

$= 1 + (x-1)\,1 + (y-1)\,0)$

$= 1 + (x-1) + \ldots = x + \ldots$

11. If $u = \frac{2x-y}{2}$, $v = y/2$

find $\frac{\partial(u,v)}{\partial(x,y)}$.

Soln:- $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$

$= \begin{vmatrix} 1 & -1/2 \\ 0 & +1/2 \end{vmatrix} = 1/2$

12. If $x = r\cos\theta$, $y = r\sin\theta$, then

find $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Soln:- $x = r\cos\theta$, $y = r\sin\theta$

$\partial x/\partial r = \cos\theta$, $\partial y/\partial r = \sin\theta$

$\partial x/\partial\theta = -r\sin\theta$; $\partial y/\partial\theta = r\cos\theta$.

$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$= r\cos^2\theta + r\sin^2\theta$

$= r$

13. If $u = x^2 + y^2$ and $x = at^2, y = 2at$, find $\frac{du}{dt}$.

Soln:

$\frac{\partial u}{\partial x} = 2x \quad \frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$

$\frac{\partial u}{\partial y} = 2y$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2x)(2at) + (2y)(2a)$$

$$= 2(at^2)(2at) + 2(2at)(2a)$$

$$= 4a^2t^3 + 8a^2t$$

$$= 4a^2t(t^2+2).$$

14. If $f(cx - az, cy - bz) = 0$ where a, b, c are constants and z is a function of x and y, find the value of $\partial z/\partial x$.

Soln: let $r = cx - az, \; s = cy - bz$

$f(r, s) = 0$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial r}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial s}\left(-b\frac{\partial z}{\partial x}\right) = 0.$$

$$c\frac{\partial f}{\partial r} - a\frac{\partial f}{\partial r}\frac{\partial z}{\partial x} - b\frac{\partial f}{\partial s}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{c\,\partial f/\partial r}{a\,\partial f/\partial r + b\,\partial f/\partial s}.$$

15. If the Cartesian Coordinates (x,y) are respresented by the polar Coordinates (r,θ) then find the Value of the Jacobian of the transformation on the unit circle.

Soln: Let $x = r\cos\theta$, $y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$J = 1$ (since it is unit circle)

16. If $x = u(1-v)$ and $y = uv$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Soln:

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}}$$

$$= \frac{1}{\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}} = \frac{1}{u}.$$

17. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the maximum possible error in the values computed for the lateral surface.

Soln: let x be the diameter and y be the height (altitude)

Surface $S = \pi xy$

$$\delta s = \frac{\partial s}{\partial x}\delta x + \frac{\partial s}{\partial y}\delta y$$

$$= \pi y\,\delta x + \pi x\,\delta y$$

$$= \pi(6\times 0.1 + 4\times 0.1)$$

$$= \pi \text{ cm.}$$

18. If $x = u(1-v)$, $y = uv$ find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln:
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u$$

19. If $x = r\cos\theta$ and $y = r\sin\theta$ then $\frac{\partial r}{\partial x}$.

Soln:

$x = r\cos\theta \Rightarrow x^2 = r^2\cos^2\theta$

$y = r\sin\theta \Rightarrow y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2$

$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}(2x)$$

$$= \frac{x}{\sqrt{x^2+y^2}} = x/r.$$

20) If $uv = x$, $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Soln: Given that

$x = uv; \quad y = u/v$

$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u,$

$\partial y/\partial u = 1/v \quad \partial y/\partial v = \frac{-u}{v^2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \frac{-2u}{v}$$

Unit - III April / May - 2019

1) Find $\frac{du}{dt}$ in terms of t if $u = x^3 + y^3$ where $x = at^2$, $y = 2at$. (2)

2) If $x = u^2 - v^2$, $y = 2uv$ find the Jacobian of x, y with respect to u and v. (2)

3) If $u = f(2x-3y, 3y-4z, 4z-2x)$, then find $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$. (8)

4) Find the shortest and the longest distances from the point $(1, 2, -1)$ to the sphere $x^2 + y^2 + z^2 = 24$. (8)

5) Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's series upto third degree terms. (8)

6) Evaluate $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values. (8)

Nov / Dec - 2020

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ when $u(x, y) = x^y + y^x$

2. If $z = x f\left(\frac{y}{x}\right)$. Then find the value of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$ using Euler's theorem.

3) (i) Let $u = 3x+2y-z$,
$v = x-2y+z$ and
$w = x(x+2y-z)$. Are u, v and w functionally related? If so, find this relationship. (8)

(ii) Find the dimensions of the rectangular box, open at the top of maximum capacity whose surface area 432 Sq. Cm. (8)

4) (i) If $z = f(x,y)$ where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, then show that
$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$. (8)

(ii) Find the Taylor's series expansion of $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ upto the second degree terms. (8)

January - 2018

1) If $x = r\cos\theta$ and $y = r\sin\theta$ then find $\frac{\partial r}{\partial x}$. (2)

2) If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$. (2)

3) If $u = (x^2+y^2+z^2)^{-1/2}$ then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ (8)

4) Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq. cm. (8)

5) Obtain the Taylor's series expansion $x^3+y^3+xy^2$ in terms of powers of $(x-1)$ and $(y-2)$ upto third degree terms (8)

6) Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$. (8)

Nov-Dec-2018

1) Find $\frac{dy}{dx}$ if $x^y + y^x = c$, where c is a constant. (2)

2) State the properties of Jacobians. (2)

3) If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, find $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$ (8)

4) Find the maxima and minima of $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. (8)

5) Find the Taylor's series expansion of function of $f(x) = \sqrt{1+x+y^2}$ in powers of $(x-1)$ and y up to second degree terms. (8)

6) Find the minimum distance from the point $(1,2,0)$ to the cone $z^2 = x^2+y^2$. (8)

Eg:-4 Find the minimum distance from the point $(1,2,0)$ to the cone $z^2 = x^2+y^2$

Soln: let $p(x,y,z)$ be any point on the cone.

The distance from the point $(1,2,0)$ to the cone is

$f = d^2 = (x-1)^2 + (y-2)^2 + (z)^2$

let $g = x^2+y^2-z^2$

let the auxiliary function F be $F = f+\lambda g$.

$F = (x-1)^2 + (y-2)^2 + z^2 + \lambda(x^2+y^2-z^2)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-1) + \lambda 2x$

$\Rightarrow x-1 = -\lambda x$

$\Rightarrow \lambda = -\left(\frac{x-1}{x}\right)$ — (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-2) + \lambda(2y)$

$\lambda = \left(-\left(\frac{y-2}{y}\right)\right)$ — (2)

$\frac{\partial f}{\partial z} = 0 \Rightarrow 2z + (-2z\lambda) = 0$

$2z = 2z\lambda$

$\boxed{\lambda = 1}$ — (3).

$\frac{x-1}{x} = \frac{y-2}{y} = -1$

$x-1 = -x$

$2x = 1 \Rightarrow x = 1/2$

$\frac{y-2}{y} = -1 \Rightarrow y-2 = -y$

$y+y = 2$

$2y = 2 \quad \boxed{y=1}$

Substitute these Values in $x^2+y^2-z^2=0$

$(\frac{1}{2})^2 + (1)^2 - z^2 = 0$

$\frac{1}{4} + 1 = z^2$

$\frac{5}{4} = z^2$

$z = \pm \frac{\sqrt{5}}{2}$

Substitute these Values in

$d^2 = (x-1)^2 + (y-2)^2 + (z)^2$

$= (\frac{1}{2}-1)^2 + (1-2)^2 + (\pm\frac{\sqrt{5}}{2})^2$

$= (-\frac{1}{2})^2 + (-1)^2 + \frac{5}{4}$

$= \frac{1}{4} + 1 + \frac{5}{4} = \frac{1}{4} + \frac{4}{4} + \frac{5}{4}$

$= \frac{10}{4} = \frac{5}{2}$

$d = \sqrt{\frac{5}{2}}$

Unit - III April / May - 2023

1. Prove $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ if $f = x^3 + y^3 + z^3 + 3xyz$.

2. If $z = x^2 + y^2$, and $x = t^2$, $y = 2at$ find $\frac{dz}{dt}$.

3. If $x = u\cos v$ and $y = u\sin v$ Prove that $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$.

4. Obtain the Taylor's series expansion of $e^x \log(1+y)$ at the origin.

5. If $u = \log\left(\frac{x^5 + y^5}{x^3 + y^3}\right)$, Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2$.

6. A rectangular box open at the top is to have volume of 32 cubic ft. find the dimensions of the box requiring least material for its construction.

Jan - 2022

1. If $u = x^3 + y^3$ where $x = a\cos t$ and $y = b\sin t$ then find $\frac{du}{dt}$.

2. If $u = \frac{2x-y}{2}$ and $v = y/z$ then find $\frac{\partial(u,v)}{\partial(x,y)}$

3. If $u = \log(\tan x + \tan y + \tan z)$ then find the value of
$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$.

4. Find the minimum value of $f(x,y) = x^2 + y^2 + 6x + 12$.

5. Expand $f(x,y) = e^x \sin y$ in terms of powers of "x" and "y" upto the third degree terms by using Taylor's series.

6) Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Nov/Dec-2022

1. Find $\frac{\partial^2 w}{\partial x \partial y}$, if $w = xy + \frac{e^y}{y^2+1}$

2. Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if $w = x^2 + y^2$, $x = r - s$ and $y = r + s$.

3. Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

4. Find the Taylor series expansion $f(x,y) = \sin x \sin y$ near the origin.

Unit - IV

1. State the fundamental theorem of Calculus

Suppose f is continuous on $[a,b]$.

(i) If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

(ii) $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f.

2. What is wrong with the equation

$$\int_{-1}^{2} \frac{4\,dx}{x^3} = \left(\frac{-2}{x^2}\right)_{-1}^{2} = \frac{3}{2}.$$

Soln: The function $f(x) = 1/x^3$ is not continuous on $[-1, 2]$. The function $f(x)$ has an infinite discontinuity at $x=0$.

$\int_{-1}^{2} \frac{4}{x^3}\,dx$ does not exist.

3. Find the general indefinite integral $\int (10x^4 - 2\sec^2 x)\,dx$.

Soln:

$$\int (10x^4 - 2\sec^2 x)\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= 10\left(\frac{x^5}{5}\right) - 2\tan x + C$$

$$= 2x^5 - 2\tan x + c$$

4. Evaluate $\int^{\infty} \frac{1}{x}\, dx$ and determine whether the integral is convergent or divergent.

Soln:

We have $\int_1^{\infty} \frac{1}{x}\, dx$

$$= \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\, dx$$

$$= \lim_{t\to\infty} \left(\log|x|\right)_1^t$$

$$= \lim_{t\to\infty} \left(\log t - \log 1\right)$$

$$= \lim_{t\to\infty} \log(t) = \infty.$$

The limit does not exist as a finite number and so the improper integral $\int_1^{\infty} \frac{1}{x}\, dx$ is divergent.

5. Evaluate $\int_4^{\infty} \frac{1}{\sqrt{x}}\, dx$.

Soln:

We have $\int_4^{\infty} \frac{1}{\sqrt{x}}\, dx = \lim_{t\to\infty} \int_4^{t} \frac{1}{\sqrt{x}}\, dx$

$$= \lim_{t\to\infty} \left(2\sqrt{x}\right)_4^t = \lim_{t\to\infty} (2\sqrt{t}) - 2\sqrt{4}$$

$$= \infty.$$

The limit does not exist as a finite number and so the improper integral $\int_{4}^{\infty} \frac{1}{\sqrt{x}} dx$ is divergent.

6. Evaluate $\int_{-1}^{1} \frac{dx}{x}$.

Soln: We first note that the given integral is improper because $f(x) = 1/x$ has the vertical asymptote $x = 0$.

We have $\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$

Since $\int_{0} \frac{dx}{x} = \lim_{t \to 0^+} \int_{t}^{1} \frac{dx}{x}$

$= \lim_{t \to 0^+} \log|x|$

$= \lim_{t \to 0^+} (\log 1 - \log|t|)$

$= \infty$.

7. Evaluate $\int \frac{x^3}{\sqrt{4+x^2}} dx$.

Soln: let us consider $x^2+4 = t$

$2x\,dx = dt$

$I = \int \frac{t-4}{2\sqrt{t}} dt \quad = \int \frac{1}{2}\sqrt{t}\,dt - 2\int t^{-1/2} dt$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} - \frac{2t^{1/2}}{1/2} + C$$

$$= \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$$

8) Evaluate $\int (\log x)^2 dx$.

Soln: Put $u = (\log x)^2 \Rightarrow du = 2\log x \frac{1}{x}dx$

$dv = dx \Rightarrow v = x$

$$I = x(\log x)^2 - \int x \cdot 2\log x \cdot \frac{1}{x} dx$$

$$= x(\log x)^2 - 2(x\log x - x) + C$$

$\left(\because \int \log x \, dx = x(\log x - 1) + C \right)$

9. Prove that the following integral by interpreting each in terms of areas $\int_0^b x\,dx = \frac{b^2-a^2}{2}$.

Soln:

$$\int_a^b x\,dx = R_1 + R_2$$

$$= a(b-a) + \frac{1}{2}(b-a)^2$$

$$= (b-a)\left(a + \frac{b-a}{2}\right)$$

$$= (b-a)\left(\frac{2a+b-a}{2}\right)$$

$$= \frac{b^2-a^2}{2}$$

10) Evaluate $\int_0^1 \tan^{-1}x \, dx$

Soln:

$u = \tan^{-1}x \quad dv = dx$

$du = \frac{1}{1+x^2}dx \quad v = x$

$$\int_0^1 \tan^{-1}(x)\, dx = \left(\tan^{-1}x\,(x)\right)_0^1 - \int_0^1 x\, d(\tan^{-1}x)$$

$$= \pi/4 - 1/2 \int_0^1 \frac{2x\, dx}{1+x^2}$$

$$= \pi/4 - 1/2 \left(\log(1+x^2)\right)_0^1$$

$$= \pi/4 - 1/2 \log 2$$

11) Evaluate $\int \frac{\cos\theta}{\sin^3\theta} d\theta$

Soln: Put $t = \sin\theta \quad dt = \cos\theta \, d\theta$

$$I = \int dt/t^3 = -1/2 \, 1/t^2$$

$$= -\frac{1}{2\sin^2\theta}$$

12) Evaluate $\int_{-1}^{1} dx/x$ if it exists

Soln: As in (Unit IV - 4)

13) What is wrong with the equation.

$$\int_{-1}^{2} 4/x^3 \, dx = \left(-2/x^2\right)_{-1}^{2} = 3/2.$$

Soln: The function $f(x) = 1/x^3$ is not continuous on $[-1, -2]$.

The function $f(x)$ has an infinite discontinuity at $x=0$

$\int_{-1}^{2} 1/x^3 \, dx$ does not exist.

14) Find the area under the parabola $y=x^2$ from 0 to 1.

Soln:

An anti derivative of $f(x)=x^2$ is $F(x) = 1/3 \, x^3$. The required area A is found by using part 2 of the fundamental theorem.

$$A = \int_0^1 x^2 \, dx$$

$$= \left(x^3/3\right)_0^1 = 1/3 - 0 = 1/3.$$

15, Find the area under the cosine curve from 0 to b where $0 \le b \le \pi/2$.

Soln: Since an anti-derivative of $f(x) = \cos x$ is $F(x) = \sin x$ we have

$$A = \int_0^b \cos x \, dx = (\sin x)_0^b$$
$$= \sin b - \sin 0$$
$$= \sin b$$
$$= \sin \pi/2$$
$$= 1.$$

16) Find the derivative of the function using the fundamental theorem of Calculus $g(x) = \int_1^x \frac{1}{t^3+1} dt$.

Soln: let $f(t) = \frac{1}{t^3+1}$ be continuous,

By the fundamental theorem of Calculus, it gives $g'(x) = \frac{1}{x^3+1}$

17) Evaluate $\int \tan^3 x \, dx$.

Soln: $\int \tan^3 x \, dx = \int \tan^2 x \tan x \, dx.$

$$= \int (\sec^2 x - 1)(\tan x) \, dx.$$

$$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx.$$

$$= \int \tan x \, d(\tan x) - \int \tan x \, dx.$$

$$= \frac{\tan^2 x}{2} - \log [\sec x] + C.$$

Unit - IV

Nov / Dec - 2020

1. Let A denote the area of the region that lies in the graph of $f(x) = \sqrt{\sin x}$ between 0 and π. Use right endpoints to find ~~the~~ an expression for A as a limit. (Do not evaluate the limit)

2) Determine whether integral $\int_1^{\infty} \frac{\ln x}{x} dx$ is Convergent or divergent. Evaluate it, if it is Convergent.

3) Prove that $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$

where m and n are positive integers.

4) Evaluate the integral

$$\int \frac{x^2 - 2x - 1}{(x-1)^2 (x^2+1)}.$$

5) Evaluate the integrals.

1) $\int x^3 \sqrt{x^2+1} \, dx$ 2) ~~$\int_0^1 \frac{1}{\sqrt{1+}}$~~

2) $\int_0^1 \frac{1}{(1+\sqrt{x})^4} \, dx$.

) Find the values of p for which the integral $\int_0^1 x^p \ln x \, dx$ Converges and evaluate the integral for those values of p.

April / May - 2019

1. Evaluate $\int_0^{\pi/2} \frac{dx}{1+\tan x}$.

2. Evaluate $\int_3^{\infty} \frac{dx}{(x-2)^{3/2}}$ and determine whether it is convergent or divergent.

3. Evaluate $\int_0^{\infty} e^{-ax} \sin bx \, dx \, (a>0)$ Using integration by parts

) Evaluate $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} \, dx$

) Evaluate $\int \frac{2x+5}{\sqrt{x^2-2x+10}} \, dx$.

) Evaluate $\int_0^{\pi/4} x \tan^2 x \, dx$.

Jan - 2018

1. What is wrong with this equation $\int_{-1}^{2} \frac{4}{x^3} dx = \left[\frac{-2}{x^2}\right]_{-1}^{2} = -\frac{3}{2}$.

2) Evaluate $\int_{4}^{\infty} \frac{1}{\sqrt{x}} dx$ and determine whether it is convergent or divergent

3) Evaluate $\int \frac{\tan x}{\sec x + \cos x} dx$

4) Evaluate $\int e^{ax} \cos bx \, dx$ using integration by parts.

5) Evaluate $\int \frac{x}{\sqrt{x^2+x+1}} dx$.

6) Evaluate $\int_{0}^{\pi/2} \cos^5 x \, dx$.

Nov/Dec - 2018

1) State the fundamental of Calculus.

2) If f is continuous and $\int_0^4 f(x)\, dx = 10$, find $\int_0^2 f(2x) dx$.

Using integration by parts, Evaluate $\int \frac{(\ln x)^2}{x^2}\, dx$

$$\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$$

5) Establish a reduction formula for $I_n = \int \sin^n x \cdot dx$. Hence, find $\int_0^{\pi/2} \sin^n x \cdot dx$.

6) For what values of p is $\int_1^{\infty} \frac{1}{x^p}\, dx$ Convergent.

Unit - IV April / May - 2022

1. Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$

2. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

3. Evaluate $\int \frac{x + \sin x}{1 + \cos x}\,dx$.

4. Use partial fraction, then evaluate $\int \frac{3x+1}{(x-1)^2 (x+3)}\,dx$.

5. Evaluate $\int_0^{\pi/4} \log(1 + \tan\theta)\,d\theta$.

6. Find the mass M and the center of mass $\bar{x}$ of a rod lying on the x axis over the interval [1, 2] whose density function is given by $\delta(x) = 2 + 3x^2$.

Jan - 2022

1. Given that $\int_0^{10} f(x)\,dx = 17$, and $\int_0^{8} f(x)\,dx = 12$ then find $\int_8^{10} f(x)\,dx$.

2. Determine whether the integral $\int_0^{\infty} \frac{dx}{x^2+4}$ is Convergent or divergent.

3. Evaluate $\int \cos^n x\,dx$ by using integration by parts.

4. Evaluate $\int \frac{dx}{\sqrt{3x-x^2-2}}$.

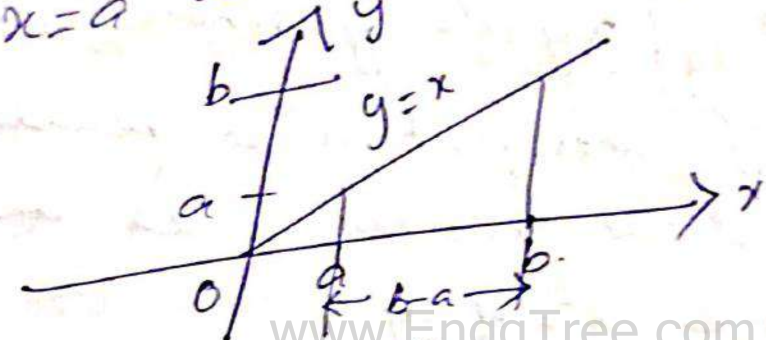
5. Evaluate $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$ by using the method of partial fractions.

6. Evaluate $\int \frac{2x+3}{x^2+x+1}\,dx$.

NOV/DEC - 2022

1. Evaluate $\int \frac{\tan x}{\sec x + \tan x}\, dx$.

2. Find the area of the region shown in the diagram given below, bounded between $x=a$ and $x=b$.



3. Evaluate $\int_0^\infty e^{-ax} \sin bx\, dx$, for $a>0$

4. Integrate $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\, dx$.

5. Evaluate $\int \frac{3x^4+3x^2-5x^2+x-1}{x^2+x-2}\, dx$

6. Integrate $\int x\sqrt{1+x-x^2}\, dx$.

Unit - V

Ex 1) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$

Soln:

$$\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dy\,dx = \int_0^1\left(x^2y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

$$= \int_0^1\left(x^2(\sqrt{x}-x) + \frac{1}{3}\left(x^{3/2} - x^3\right)\right)dx$$

$$= \int_0^1\left(x^{3/2} - x^3 + \frac{x^{3/2}}{3} - \frac{1}{3}x^3\right)dx$$

$$= \int_0^1\left(\frac{4}{3}x^{3/2+1} - \frac{4}{3}x^3\right)dx$$

$$= \left(\frac{4}{3}\,\frac{x^{5/2}}{5}\times 2 - \frac{4}{3}\,\frac{x^4}{4}\right)_0^1$$

$$= \left(\frac{4\times 2}{3\times 5} - \frac{4}{3\times 4}\right)$$

$$= \left(\frac{8}{15} - \frac{1}{3}\right)$$

$$= \cancel{\frac{32-30}{60} = \frac{2}{60}} \qquad = \frac{24-15}{45}$$

$$= \frac{9}{45} = \frac{3}{15}$$

$$= \frac{3}{15}.$$

2) Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} r^2\,dr\,d\theta$

Soln:

$$\int_{-\pi/2}^{\pi/2}\left(\frac{r^3}{3}\right)_0^{2\cos\theta} d\theta = \frac{1}{3}\int_{-\pi/2}^{\pi/2}(8\cos^3\theta - 0)\,d\theta$$

$$= \frac{8}{3}\times 2\int_0^{\pi/2}\cos^3\theta\,d\theta$$

$$= \frac{8}{3}\times 2\times \frac{2}{3}\cdot 1$$

$$= \frac{32}{9}.$$

3) Evaluate $\int_0^1\int_1^2 x(x+y)\,dy\,dx$.

Soln:

$$\int_0^1\int_1^2 (x^2+xy)\,dy\,dx$$

$$= \int_0^1\left(x^2y + \frac{xy^2}{2}\right)_1^2 dx$$

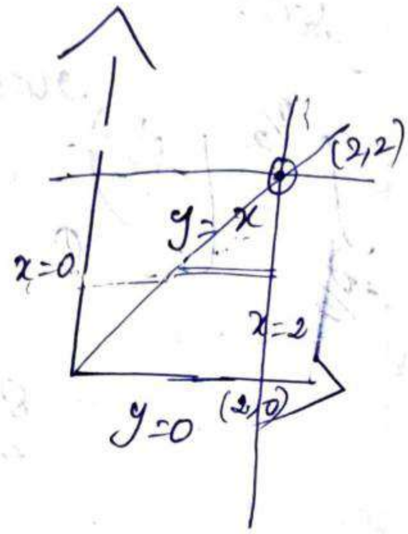
$$= \int_0^1 x^2(2) + \frac{x(4)}{2} - x^2 - \frac{x}{2}$$

$$= \int_0^1\left(x^2 + \frac{3x}{2}\right)dx = \left(\frac{x^3}{3} + \frac{3x^2}{4}\right)_0^1$$

$$= \frac{1}{3} + \frac{3}{4} = \frac{4+9}{12} = \frac{13}{12}$$

4) Change the order of integration in $\int_0^2 \int_0^x f(x,y)\, dy\, dx$.

Soln:



x limit

$x = 0$ to $x = 2$

y limit

$y = 0$ to $y = x$

$$\int_0^2 \int_y^2 f(x,y)\, dx\, dy.$$

5) Change the order of integration of $\int_0^3 \int_1^{\sqrt{4-y}} f(x,y)\, dx\, dy$.

Soln:

Here the region of integration is $x=1$, $x=\sqrt{4-y}$, $y=0$, $y=3$.

After changing the order of integration, the first integration is with respect to y, and the second integration is with respect to x.

$\cancel{x = \sqrt{4-y}}$

$\cancel{x^2 = 4-y}$

$x = 1$

$x = \sqrt{4-y}$

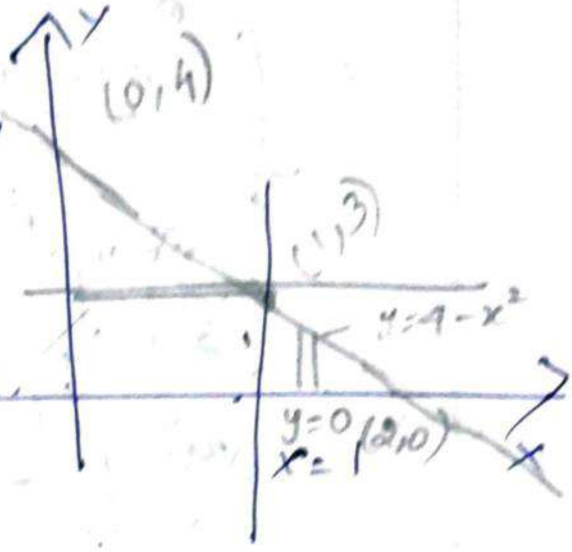
x	0	1
y	4	3

$x^2 = 4-y$

$y = 4-x^2$

$$I = \int_0^3 \int_1^{\sqrt{4-y}} f(x,y)\,dx\,dy$$

$$= \int_{x=1}^{2} \int_{y=0}^{4-x^2} f(x,y)\,dy\,dx$$



6) Evaluate $\iint r^2 \sin\theta \, dr\, d\theta$ where R is the semicircle $r = 2a\cos\theta$ above the initial line.

Soln:

$$I = \int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

Put $z = \cos\theta$, $dz = -\sin\theta \, d\theta$

If $\theta = 0$, $z = 1$

$\theta = \pi/2$, $z = 0$.

$$\therefore \int_1^0 \int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_1^0 \int_0^{2a\cos\theta} r^2 \sin\theta \, dr\, d\theta$$

$$= \int_1^0 \left(\frac{r^3}{3}\right)_0^{2a\cos\theta} \sin\theta \, d\theta$$

$$= \int_1^0 \frac{8a^3}{3} \cos^3\theta \sin\theta \, d\theta$$

$$= \frac{8a^3}{3} \int_1^0 z^3 (-dz)$$

$$= \frac{8a^3}{3} \int_0^1 z^3 dz$$

$$= \frac{8a^3}{3} \left(\frac{z^4}{4}\right)$$

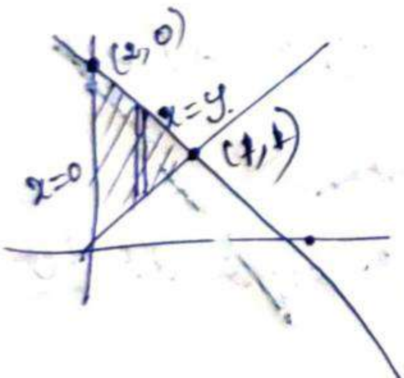
$$= \frac{a^3 \times 8}{3} \left(\frac{1}{4}\right)$$

$$= \frac{2a^3}{3}.$$

7) Find the area bounded by the curves $x = y$, $x + y = 2$ and $x = 0$ using double integration.

Soln:

$$\text{Area} = \int_0^1 \int_x^{2-x} dy\, dx$$



$y = x, \quad y = 2 - x$

x	0	2	1
y	2	0	1

$$= \int_0^1 (y)_x^{2-x} dx$$

$$= \int_0^1 (2 - x - x)\, dx$$

$$= \int_0^1 (2-2x)\,dx$$

$$= 2\left(x - \frac{x^2}{2}\right)_0^1$$

$$= 2\left(1-\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1$$

8) Evaluate the improper integral $\int_2^3 \frac{dx}{\sqrt{3-x}}$ if possible.

Soln:

$$\int_2^3 \frac{dx}{\sqrt{3-x}} = \cancel{\lim_{\varepsilon\to 0}\int_2^3 \frac{dx}{\sqrt{3-x}}}$$

$$\cancel{= \lim_{\varepsilon\to 0}\left(-2\right.}$$

$$= \int_2^3 (3-x)^{-1/2}\,dx$$

$$= \left[2\,\frac{\sqrt{3-x}}{(-1)}\right]_2^3$$

$$= -\left(2(0) - 2\right)$$

$$= 2.$$

9) Evaluate $\int_{x=0}^{1}\int_{y=0}^{2} (x+y)\,dy\,dx$.

Soln:

$$I = \int_{x=0}^{1} \int_{y=0}^{2} (x+y)\, dy\, dx$$

$$= \int_0^1 \left(xy + \frac{y^2}{2} \right)_0^2 dx$$

$$= \int_0^1 \left(2x + \frac{4}{2} \right) dx$$

$$= \left(\frac{2x^2}{2} + \frac{4x}{2} \right)_0^1$$

$$= \left(x^2 + 2x \right)_0^1$$

$$= (1+2) = 3$$

(c) Describe the solid region whose volume is given by the following triple integral

$$\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{1} dz\, dy\, dx$$

(Do not evaluate the integral)

Soln:

$$R = \left\{ (x,y,z) \,/\, -1 \le x \le 1,\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\ 0 \le z \le 1 \right\}$$

Cylinder of height 1 unit with base curve $x^2+y^2=1$

11) Find the value of

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy.$$

Soln:

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y} (x)_0^y \, dy \right)$$

$$= \int_0^\infty \frac{e^{-y}}{y} (y)\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

$$= \left(-(0-1)\right) = 1$$

12) Find the limits of integration in the double integration $\iint_R f(x,y)\, dx\, dy$ where R is in the first quadrant and bounded by $x=1$, $y=0$, $y^2=4x$.

Soln: Region of integration

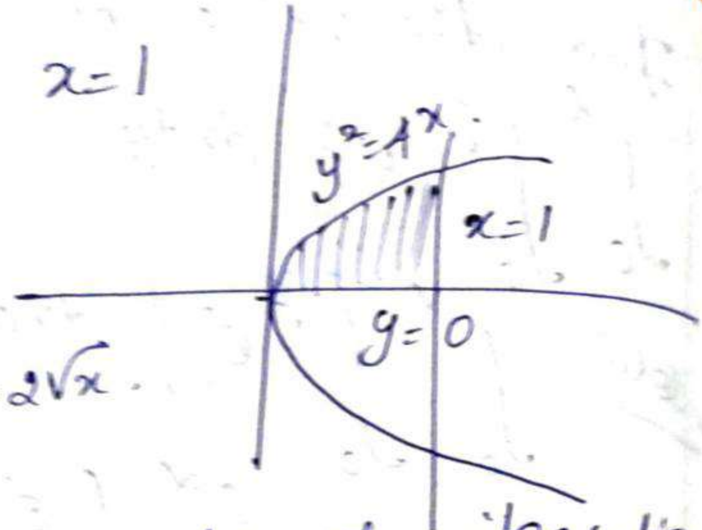
$y=0$, $y^2=4x$, $y=0$.

The limits of integration are

$x=0$ to $x=1$

~~the~~.

$y=0$ to $y=2\sqrt{x}$.



13) Change the order of integration in $\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$

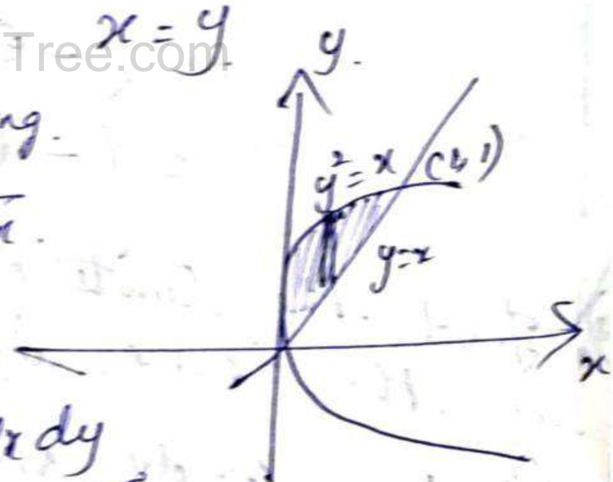
Soln:

The limits are

$x=y^2 \quad x=y$

After changing

$y=x, \quad y=\sqrt{x}$.



$$\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dy\,dx.$$

14. Evaluate $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\,dx\,dy$

Soln:

$$= \int_1^{\ln 8} e^y \left(e^x\right)_1^{\ln y} dy$$

$$= \int_1^{\ln 8} e^y (y-1)\, dy$$

$$= \int_1^{\ln 8} e^y (y-1)\, dy$$

$$= \int_1^{\ln 8} y e^y - \int_1^{\ln 8} e^y dy$$

$$= \left(y e^y\right)_1^{\ln 8} - \int_1^{\ln 8} e^y dy \quad - \left(e^{\ln 8} - e\right)$$

$$= (\ln 8)\, 8 - e - 8 - e$$

$$- 8 + e$$

$$= 8 \ln 8 - 16 + e$$

15) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\, dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2}\right) dx$$

$$= \left(\frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2}\right)_0^a$$

$$= \frac{a^2}{2} \frac{\pi}{2} = \frac{\pi a^2}{4}$$

16) Evaluate $\int_0^1 \int_0^{1-x} y\, dy\, dx$

Soln:

$$\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx = \int_0^1 \left(\frac{(1-x)^2}{2}\right) dx$$

$$= \left(\frac{(1-x)^3}{3(2)(-1)}\right)_0^1$$

$$= \left(0 - \left(\frac{-1}{6}\right)\right)$$

$$= \frac{1}{6}$$

17) Evaluate $\int_0^1 \int_0^2 \int_0^3 (xyz)\, dz\, dy\, dx$

$$= \int_0^1 x\, dx \int_0^2 y\, dy \int_0^3 z\, dz$$

$$= \frac{1}{2} \times \frac{4}{2} \times \frac{9}{2} = \frac{9}{2}$$

18) Evaluate $\int_0^2 \int_1^3 \int_1^2 (xy^2z)\, dz\, dy\, dx$

$$= \left(\frac{x^2}{2}\right)_0^2 \left(\frac{y^3}{3}\right)_1^3 \left(\frac{z^2}{2}\right)_1^2$$

$$= \left(\frac{4}{2}\right)\left(\frac{27}{3}\right)\left(\frac{4}{2}\right)$$

$$= 2 \times 9 \times 2 = 36$$

19) $\int_{-1}^{2}\int_{x}^{x+2} dx\, dy$

$= \int_{-1}^{2} (y)_{x}^{x+2}\, dx$

$= \int_{-1}^{2} (x+2-x)\, dx$

$= 2\int_{-1}^{2} dx$

$= 2\,(x)_{-1}^{2}$

$= 2\,(2+1)$

$= 6.$

20) Evaluate $\int_{0}^{\pi/2}\int_{0}^{\sin\theta} r\, dr\, d\theta.$

Soln:

$\int_{0}^{\pi/2}\int_{0}^{\sin\theta} r\, d\theta\, dr = \int_{0}^{\pi/2}\left(\frac{r^2}{2}\right)_{0}^{\sin\theta} d\theta$

$= \frac{1}{2}\int_{0}^{\pi/2} \sin^2\theta\, d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)$

$= \pi/8.$

Unit - V

Multiple Integrals

Nov-Dec-2020

1. Find the area of a circle of radius "a" by double integration in polar coordinates.

2. Evaluate $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} xy \, dx\, dy\, dz$.

3. Change the order of integration in $\int_{0}^{1}\int_{y}^{2-y} xy \, dx\, dy$ and then evaluate it.

4) Evaluate $\iiint_V xyz \, dx\, dy\, dz$ where V is the volume of the positive octant of the sphere $x^2+y^2+z^2=1$ by transforming to spherical polar coordinates.

5) Evaluate $\iint_D xy\sqrt{1-x-y} \, dx\, dy$, where D is the region bounded by $x=0$, $y=0$ and $x+y=1$, using the transformation $x+y=u$, $y=uv$

6) find the volume of the cylinder bounded by $x^2+y^2=4$ and the planes $y+z=4$ and $z=0$ using triple integral.

April / May - 2019

1. Evaluate $\int_1^a \int_2^b \frac{dx\,dy}{xy}$

2. Find the limits of integration $\iint_R f(x,y)\,dx\,dy$ where R is the triangle bounded by $x=0, y=0, x+y=2$.

3. Change of the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and then evaluate it.

4. Evaluate by changing to polar coordinates $\int_0^a \int_y^a \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}}$

5. Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant bounded by $x/a + y/b = 1$.

6) Find the value of $\iiint xyz\,dz\,dy\,dx$ through the positive spherical octant for which $x^2+y^2+z^2 \le a^2$

Jan - 2018

1. Find the value of $\int_0^{\infty}\int_0^{y}\left(\frac{e^{-y}}{y}\right)dx\,dy$

2. Find the limits of integration in the double integral $\iint_R f(x,y)\,dx\,dy$ where R is in the first quadrant and bounded by $x=1$, $y=0$, $y^2=4x$.

3. Change the order of integration for the given integral $\int_0^a \int_0^{2\sqrt{ax}} (x^2)\,dy\,dx$

4. Evaluate by changing to polar coordinates $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

5. Evaluate $\iiint (xyz)\,dx\,dy\,dz$ over the first octant of $x^2+y^2+z^2=a^2$.

(6) Using double integral, find the area bounded by $y=x$ and $y=x^2$.

Nov-Dec-2018

1. $\int_0^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$

2. Change the order of integration in $\int_0^1\int_{y^2}^{y} f(x,y)\,dx\,dy$.

3. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.

4. Express $\int_0^a\int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy$ in polar coordinates and then evaluate it.

5. find the area bounded by the parabola $y^2=4-x$ and $y^2=x$.

6. Evaluate $\iiint_V dx\,dy\,dz$, where V is the finite region of space (tetrahedron) bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+3y+4z=12$.

Unit-V January - 2022

1. Evaluate $\int_{-\pi/2}^{\pi/2}\int_{0}^{2\cos\theta} d\theta\, ds.$

2. $\int_0^1\int_0^2\int_0^3 (xy^2z)\, dx\, dy\, dz.$

3. Evaluate $\iint xy\, dx\, dy$ where the region of integration is bounded by the lines x-axis, $x=2a$ and the curve $x^2=4ay$.

4. Change the order of integration in $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\, dy\, dx$ and hence evaluate it.

5. Evaluate $\int_0^a\int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ by change into polar coordinates.

6. Evaluate $\int_0^{2a}\int_0^{x}\int_y^{x} (xyz)\, dz\, dy\, dx$

April / May 2022

1. Evaluate $\int_1^2 \int_1^3 xy^2 \, dx \, dy$

2. $\int_0^1 \int_0^2 \int_0^3 xyz \, dx \, dy \, dz$.

3. Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$ and evaluate the same.

4. Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

5. Using polar coordinates $\iint_R e^{x^2+y^2} dy dx$ where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$.

6. Calculate the volume of the solid bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$

Nov/Dec 2022

1. Sketch the region of integration in $\int_0^1 \int_x^1 f(x,y)\, dy\, dx$

2. Change the Cartesian integral $\int_0^6 \int_0^y x\, dx\, dy$ into an equivalent polar integral.

3. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate.

4. Find the area of the region inside the Cardioid $r = a(1+\cos\theta)$ and outside the Circle $r = a$.

5. Find the volume of the region by the paraboloid $z = x^2+y^2$ and the plane $z = 4$.